Math Wrangle Problems
Joint Mathematics Meetings, San Antonio

American Mathematics Competitions

January 11, 2015

1. Find the sum of all positive two-digit integers that are divisible by each of their digits.

   Let a represent the tens digit and b the units digit of an integer with the required property. Then $10a + b$ must be divisible by both a and b. It follows that b must be divisible by a, and that $10a$ must be divisible by b. The former condition requires that $b = ka$ for some positive integer k, and the latter condition implies that $k = 1$ or $k = 2$ or $k = 5$. Thus the requested two-digit numbers are 11, 22, 33, $\ldots$, 99, 12, 24, 36, 48, and 15. Their sum is $11 \cdot 45 + 12 \cdot 10 + 15 = 630$.

2. An equilateral triangle is inscribed in the ellipse whose equation is $x^2 + 4y^2 = 4$. One vertex of the triangle is $(0, 1)$, one altitude is contained in the y-axis. Find the length of each side of this equilateral triangle.

   Let the other two vertices of the triangle be (x, y) and $(-x, y)$, with $x > 0$. Then the line through $(0, 1)$ and (x, y) forms a 120-degree angle with the positive x-axis, and its slope is $\tan(120°) = -\sqrt{3}$. Therefore, the line's equation is $y = -\sqrt{3}x + 1$. Substituting this into the equation of the ellipse and simplifying yields

   $$13x^2 - 8\sqrt{3}x = 0 \qquad \text{or} \qquad x = \frac{8\sqrt{3}}{13}.$$

   The triangle has sides of length $2x = (16\sqrt{3})/13 = \sqrt{768/169}$.

3. A fair die is rolled four times. Find the probability that each of the final three rolls is at least as large as the roll preceding it. Any particular outcome of the four rolls has probability $1/6^4$.

   Given the values of four rolls, there is exactly one order that satisfies the requirement. It therefore suffices to count all the sets of values that could be produced by four rolls, allowing duplicate values. This is equivalent to counting the number of ways to put four balls into six boxes labeled 1 through 6. By thinking of 4 balls and 5 dividers to separate the six boxes, this can be seen to be $\binom{9}{4} = 126$. The requested probability is thus $126/6^4 = 7/72$.

   OR

   Let a_1, a_2, a_3, and a_4 be the sequence of values rolled, and consider the difference between the last and the first: If $a_4 - a_1 = 0$, then there is 1 possibility for a_2 and a_3, and 6 possibilities for a_1 and a_4. If $a_4 - a_1 = 1$, then there are 3 possibilities for a_2 and a_3, and 5 possibilities for a_1 and a_4. In general, if $a_4 - a_1 = k$, then there are $6 - k$ possibilities for a_1 and a_4, while the number of possibilities for a_2 and a_3 is the same as the number of sets of 2 elements, with repetition allowed, that can be chosen from a set of $k+1$ elements. This is equal to the number of ways to put 2 balls in $k+1$ boxes, or $\binom{k+2}{2}$. Thus there are $\sum_{k=0}^{5} \binom{k+2}{2}(6-k) = 126$ sequences of the type requested, so the probability is $126/6^4 = 7/72$.

4. Three of the vertices of a cube are $P = (7, 12, 10)$, $Q = (8, 8, 1)$, and $R = (11, 3, 9)$. What is the surface area of the cube?

   Notice that PQR is an equilateral triangle, because $PQ = QR = RP = 7\sqrt{2}$. This implies that each edge of the cube is 7 units long. Hence the surface area of the cube is $6(7^2) = 294$.

5. Find the integer that is closest to $1000 \displaystyle\sum_{n=3}^{10000} \frac{1}{n^2 - 4}$.

   Because $\dfrac{1}{n^2 - 4} = \dfrac{1}{4}\left(\dfrac{1}{n-2} - \dfrac{1}{n+2}\right)$, the series telescopes, and it follows that

$$1000 \sum_{n=3}^{10000} \frac{1}{n^2-4} = 250\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{9999}-\frac{1}{10000}-\frac{1}{10001}-\frac{1}{10002}\right)$$
$$= 250+125+\frac{250}{3}+\frac{250}{4}-\frac{250}{9999}-\frac{250}{10000}-\frac{250}{10001}-\frac{250}{10002}$$
$$= 520+\frac{5}{6}-r$$

where the positive number r is less than $1/3$. Thus the requested integer is 521.

6. Let $\mathcal{S}$ be the set $\{1, 2, 3, \ldots, 10\}$. Let n be the number of sets of two non-empty disjoint subsets of $\mathcal{S}$. (*Disjoint sets* are defined as sets that have no common elements.) What is n?

   Let k be the number of elements of $\mathcal{S}$, and let A and B be two empty jars into which elements of $\mathcal{S}$ will be placed to create two disjoint subsets. For each element x in $\mathcal{S}$, there are three possibilities: place x in A, place x in B, or place x in neither A nor B. Thus the number of ordered pairs of disjoint subsets (A, B) is 3^k. However, this counts the pairs where A or B is empty. Note that for A to be empty, there are two possibilities for each element x in $\mathcal{S}$: place x in B, or do not place x in B. The number of pairs for which A or B is empty is thus $2^k + 2^k - 1 = 2^{k+1} - 1$. Since interchanging A and B does not yield a different set of subsets, there are $\frac{1}{2}(3^k - 2^{k+1} + 1) = \frac{1}{2}(3^k + 1) - 2^k$ sets. When $k = 10$, $n = \frac{3^{10}+1}{2} - 2^{10} = 28501$.