



# SOME MATH CIRCLE QUESTIONS

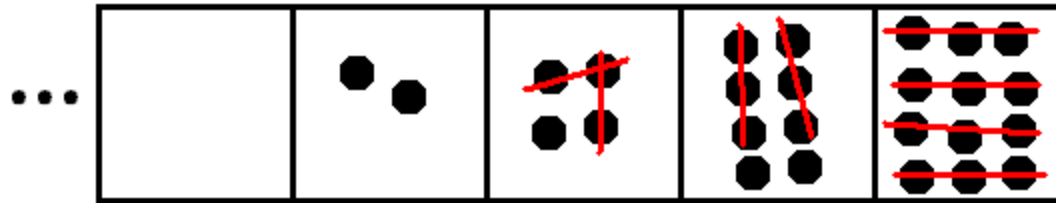


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**MAA, Mathematician In Residence**



2 ← 3

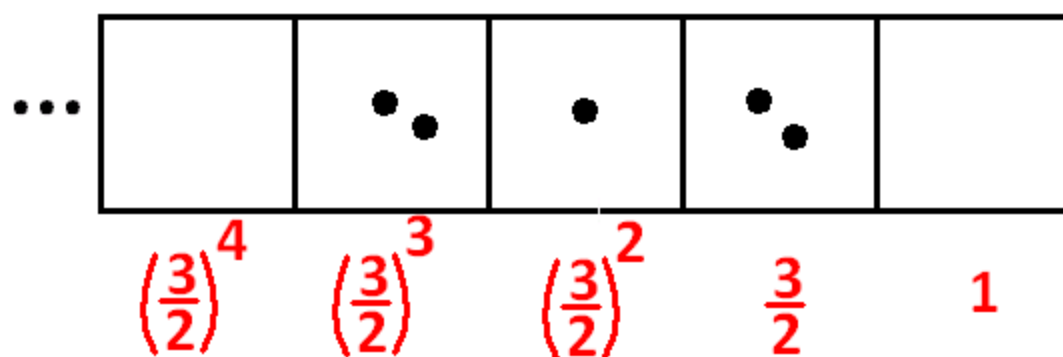


1 → 1	7 → 211
2 → 2	8 → 212
3 → 20	9 → 2100
4 → 21	10 → 2101
5 → 22	11 → 2102
6 → 210	12 → 2120

**All well and good, but what is this machine doing?**



2 ← 3



$$2 \times \left(\frac{3}{2}\right)^3 + 1 \times \left(\frac{3}{2}\right)^2 + 2 \times \left(\frac{3}{2}\right) + 0 \times 1 = 2 \times \frac{27}{8} + 1 \times \frac{9}{4} + 2 \times \frac{3}{2} = \frac{27}{4} + \frac{9}{4} + 3 = 12$$



**Every positive integer can be represented as a combination of powers of  $3/2$  using the coefficients 0, 1 and 2. **UNIQUELY!****

### **An infinite number of asides:**

**Every positive integer can be uniquely expressed in terms of powers of  $17/5$  using the coefficients 0, 1, 2, ...,16.**

**Every positive integer can be uniquely expressed in terms of powers of  $103/98$  using the coefficients 0, 1, 2, ...,102.**

**Every positive integer can be uniquely expressed in terms of powers of  $988762/133$  using the coefficients 0, 1, 2, ...,988721.**



0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2102220
2101	21202	212020	2102221

After a small start, all representations begin with 2.

All (eventually) begin with 21? First three digits eventually stable?  
First N digits?



0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2102220
2101	21202	212020	2102221

Final digits cycle 0, 1, 2.

Multiples of three end in 0.

Divisibility Rule for Three:

*N is divisible by 3 iff Final digit is zero*



0			
2	2120	21221	212022
21	2122	210110	212211
210	21011	210112	2101100
212	21200	212001	2101102
2101	21202	212020	2102221

**Is there a divisibility rule for 2?**

**How does one tell if a number is even?**



0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2102220
2101	21202	212020	2102221

For that matter ... How do you tell if a given representation is an integer?

**Is 2102020210110021222021011 an integer?**

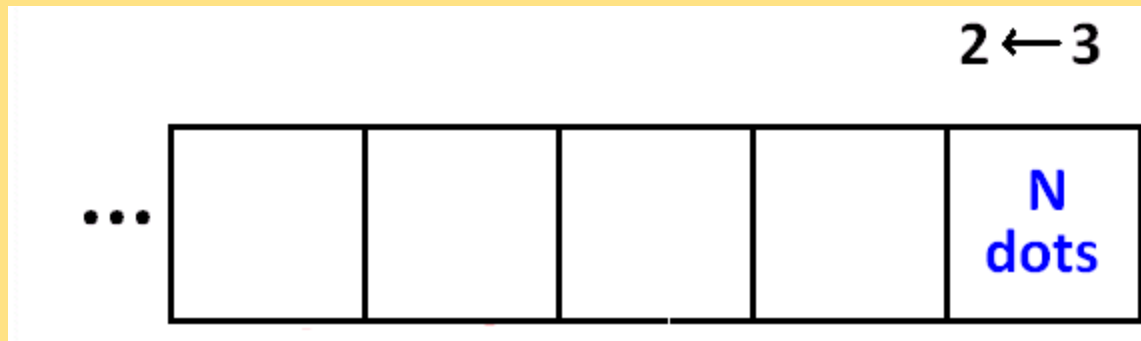




	0	3	6	9	15	24	...
0							
1	2102	21220	212021				
2	2120	21221	212022				
20	2121	21222	212210				
21	2122	210110	212211				
22	21010	210111	212212				
210	21011	210112	2101100				
211	21012	212000	2101101				
212	21200	212001	2101102				
2100	21201	212002	2102220				
2101	21202	212020	2102221				

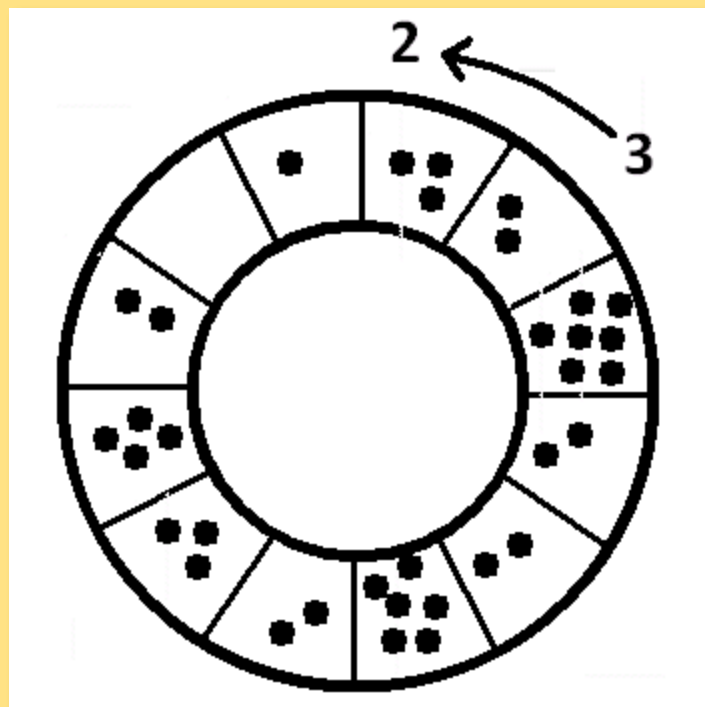
**Fibonacci-esque?**

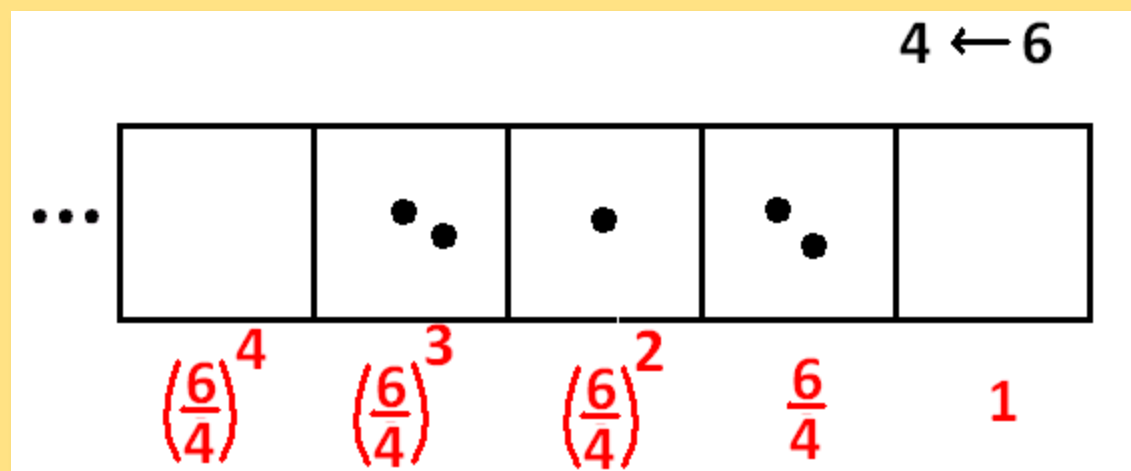
**0, 3, 6, 9, 15, 24, 36, 54, 81, 123, 186, 279, 420, 630, ...**



**CHALLENGE:** Prove that the order in which one conducts the explosions does not matter. The final distribution of dots is sure to be the same for all choices.

(The proof relies on there being a “boundary” at the right end.)







<b>0</b>	<b>0</b>				
1	1	2102	45	21220	4203
2	2	2120	<b>420</b>	21221	4204
<b>20</b>	3	2121	421	21222	4205
21	4	2122	422	<b>210110</b>	4240
22	5	<b>21010</b>	423	210111	4241
<b>210</b>	<b>40</b>	21011	424	210112	4242
211	41	21012	425	212000	4243
212	42	21200	<b>4200</b>	212001	4244
<b>2100</b>	43	21201	4201	212002	4245
2101	44	21202	4202	212020	<b>44020</b>
				212021	44021
				212022	44022
				212210	44023
				212211	42044
				212212	42045
				<b>2101100</b>	42420
				2101101	42421
				2101102	42422
				2102220	42423
				2102221	42424

0, 3, 6, 9, 15, 24, 36, 54, ...

0, 6, 12, 18, 30, 48, 72, 108, ...

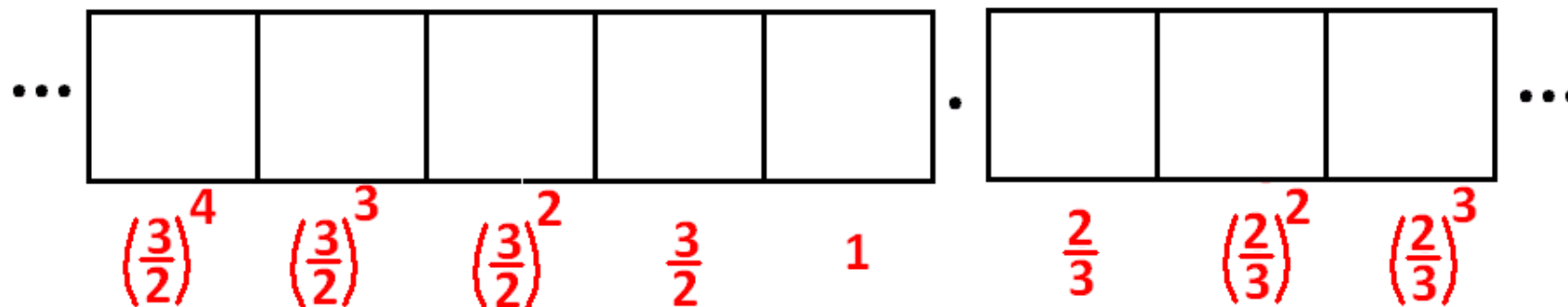


## Number of Explosions

0	0				
1	0	2102	6	21220	14
2	0	2120	7	21221	14
20	1	2121	7	21222	14
21	1	2122	7	210110	19
22	1	21010	11	210111	19
210	3	21011	11	210112	19
211	3	21012	11	212000	22
212	3	21200	13	212001	22
2100	6	21201	13	212002	22
2101	6	21202	13	212020	23
				212021	23
				212022	23
				212210	25
				212211	25
				212212	25
				2101100	31
				2101101	31
				2101102	31
				2102220	30
				2102221	30



$2 \leftarrow 3$



$$1/2 = 0.01\ 000001\ 001\ 001\ 01\ 0000000001\ 0000001\ 0001\ 0000001\ 001\ 001\ 01\ \dots$$

$1/2 =$  0. 01 000001001 001010000000001000000100001 000001 001001 01 000000000010001001 0000001001 00001 00001000  
 0001 0001 0001 000001001 000010000000100000001 0010000010001 00001001 0001 000001000001001 001001 001 0000100  
 01 00000000001000001 000001 00101000000101000001001 0001001 0001 0001 001000000001 00000010000000001 00000  
 01 0000000001001 000000010000010001001001 00001000101000000000000010000000001 0001 000000100000001 0001 0  
 01 001000100001 0001001 000000100001 001 001001 00100000100100000001 001000000010001001 001001 0001000100  
 100000100001 00000001...



## REFERENCES:

**COME TO EXPLODING DOTS WORKSHOP!**



***TOMORROW***  
***6:00 p.m. - 7:50 p.m.***  
***ROOM 25***



MAA Curriculum Inspirations: [www.maa.org/node/129058](http://www.maa.org/node/129058)

SIGMAA MCST: [sigmaa.maa.org/mcst](http://sigmaa.maa.org/mcst)

Tanton Stuff: [www.jamestanton.com](http://www.jamestanton.com)  
[www.gdaymath.com](http://www.gdaymath.com)

**ALSO ... Just google “EXPLODING DOTS”**