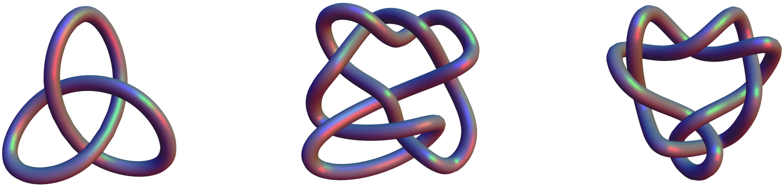
Knotted Mathematics for Elementary-Aged Students

Mark Hughes

January 11, 2018

Knots

Knots: Embeddings of S^1 into $\mathbb{R}^3$, considered up to isotopy.



Knots are fun to study, and are useful in

- Molecular biology (DNA)
- Polymer chemistry
- Physics
- Geometry

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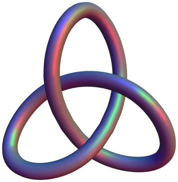
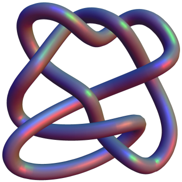
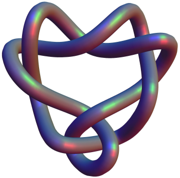






Purpose: Use knots to introduce students to the concepts of equivalence between mathematical objects, and how to use invariants to distinguish inequivalent objects.

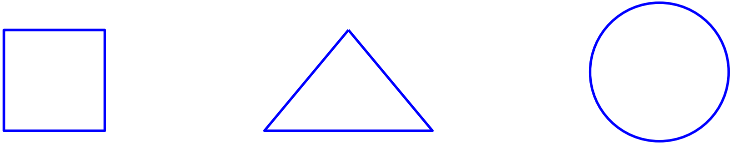
Audience: Elementary-aged students (topics can be scaled-up to older students).

Both the Same and Different

Similarities: birds, feathers, beaks, can fly, etc.

Differences: white head vs. green head, talons vs. webbed feet, hooked beak vs. flat bill, etc.

Which Shape is Which?



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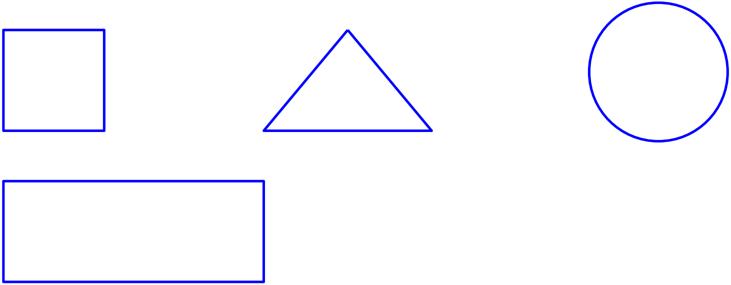

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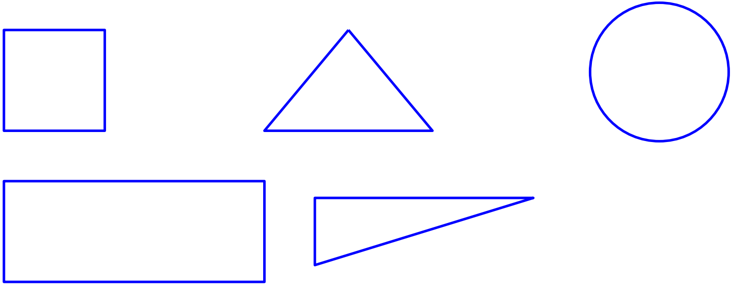

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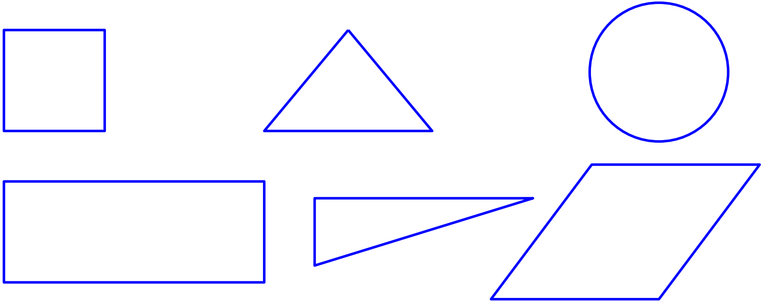

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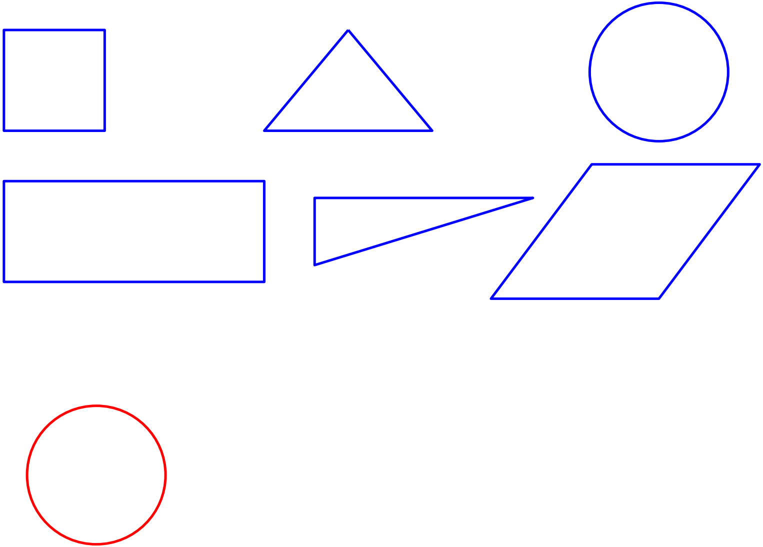

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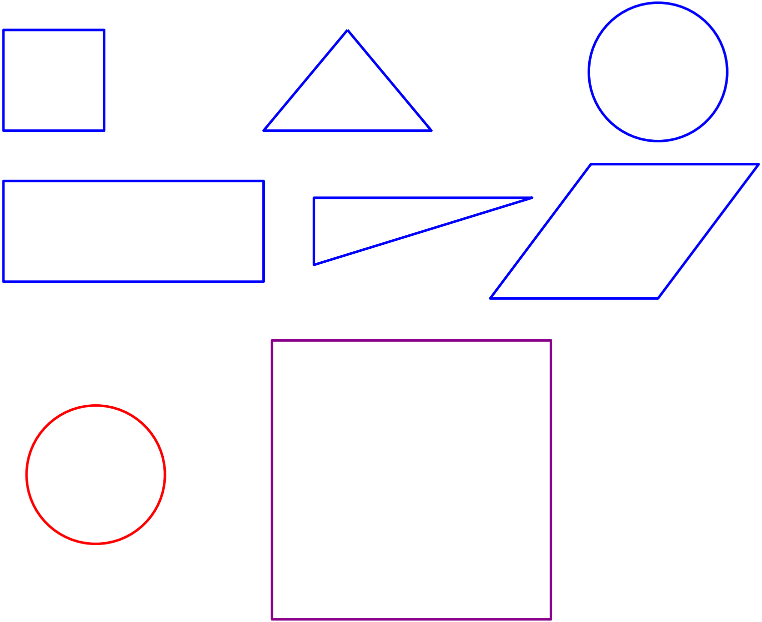

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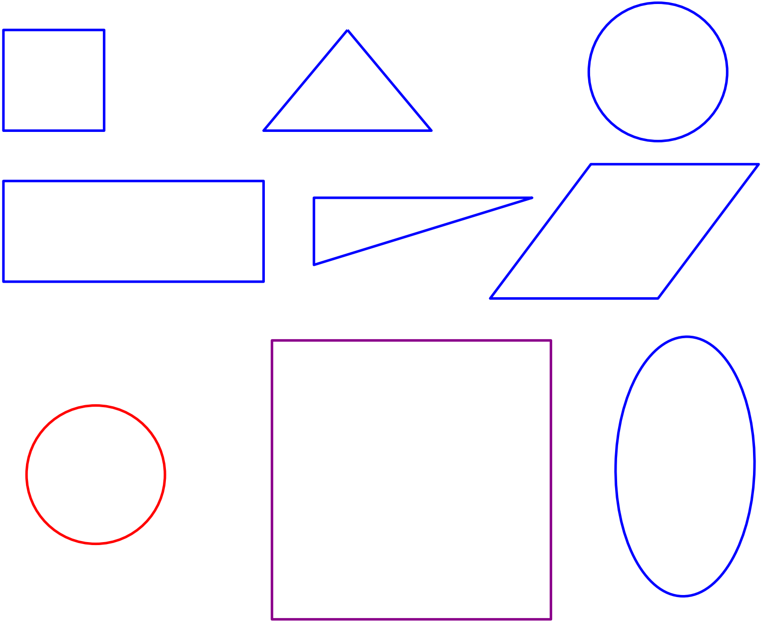
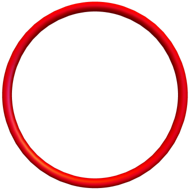
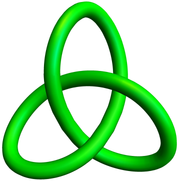


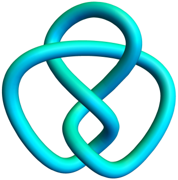
Your Friendly Neighborhood Knots



THE UNKNOT



THE TREFOIL



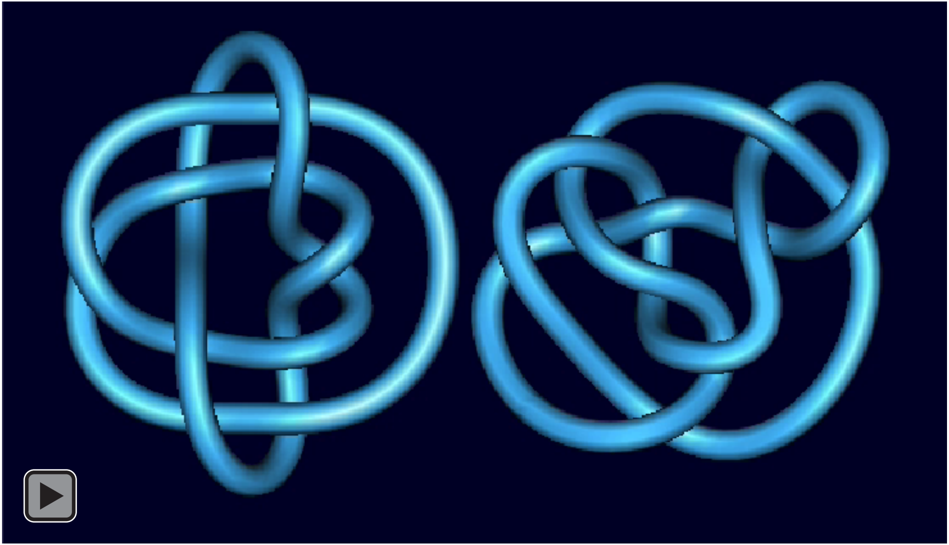
THE FIGURE 8

How is the unknot different from the trefoil?

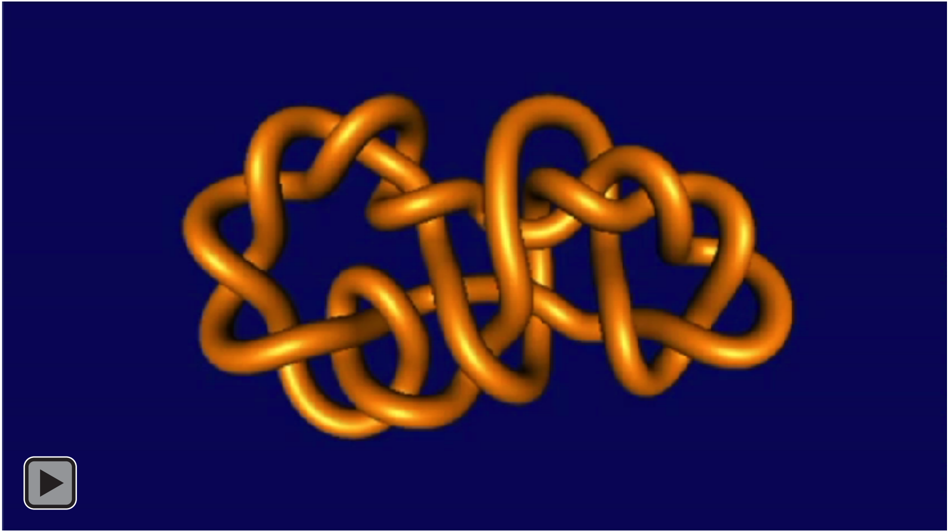
How does the trefoil differ from the figure 8 knot?

Does your answer depend on the way that we've drawn the knots?

When Are Two Knots “The Same”?



Tricky Knots



https://www.youtube.com/watch?v=UmF0-Tz1oWc

How Do We Tell Knots Apart?

Coloring Knot Diagrams

Suppose we have a knot diagram.

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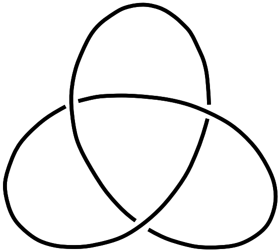

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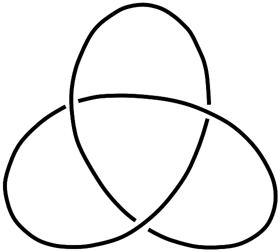

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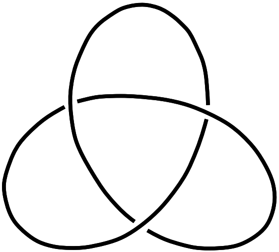

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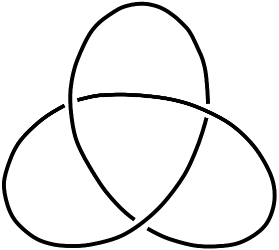

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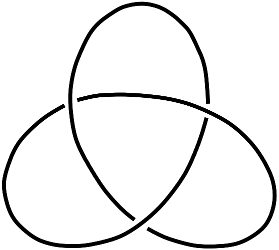

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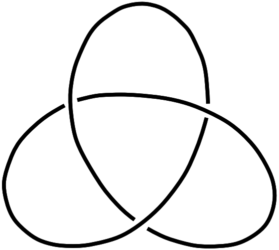

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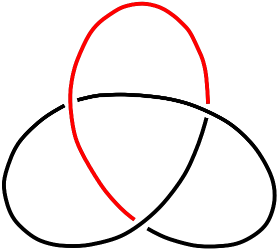

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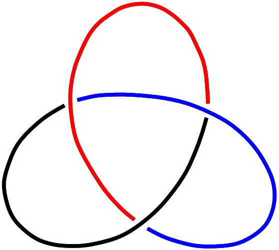

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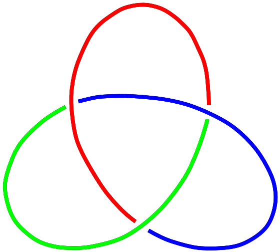

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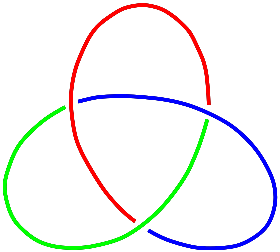
Coloring Knot Diagrams

Since the trefoil knot can be colored according to these rules, we say that it is 3-colorable.



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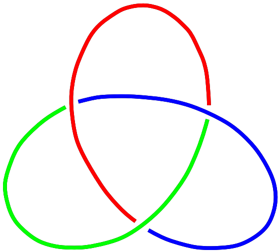

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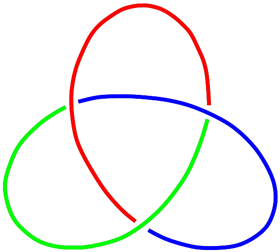

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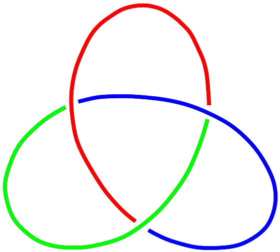

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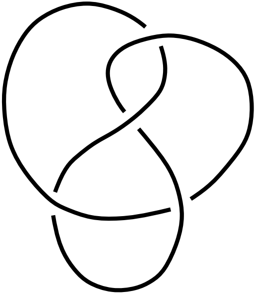

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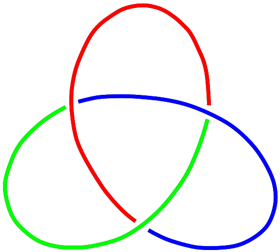

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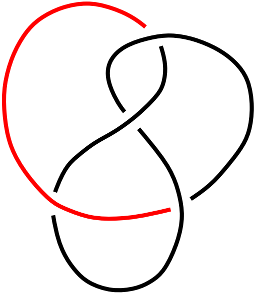

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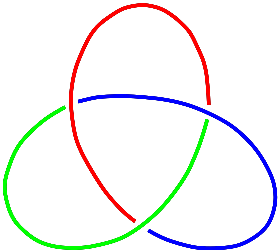

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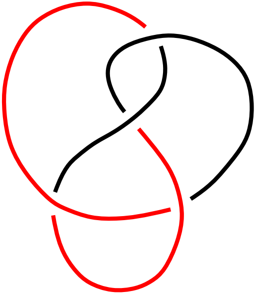

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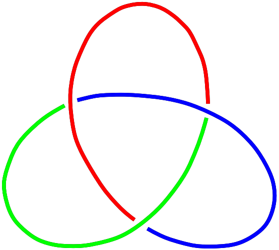

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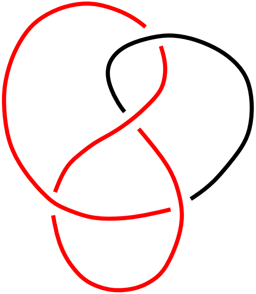

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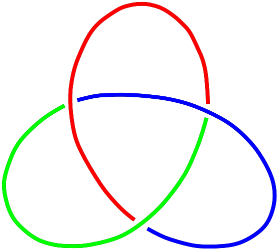

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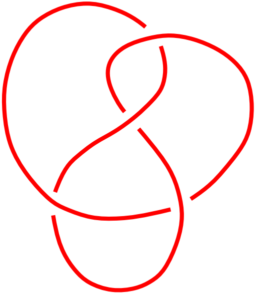

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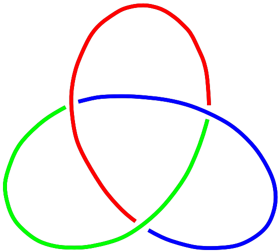

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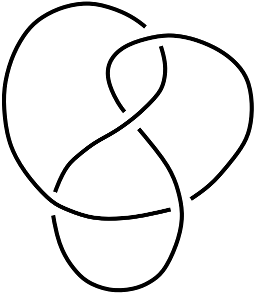

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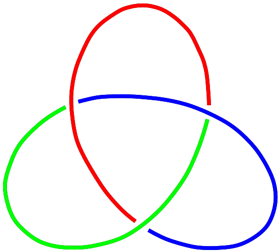

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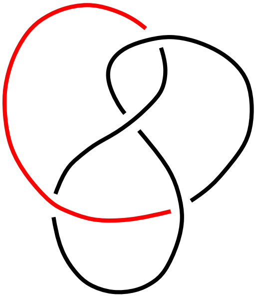

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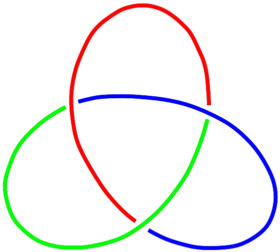

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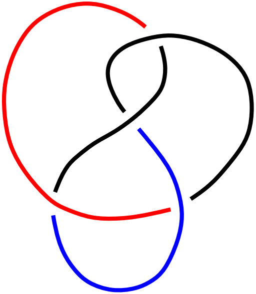

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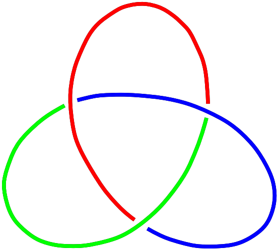

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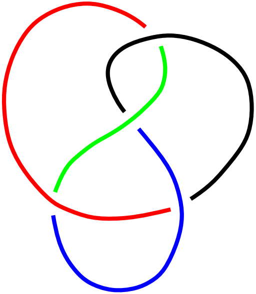

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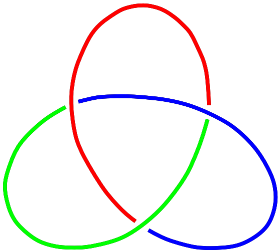

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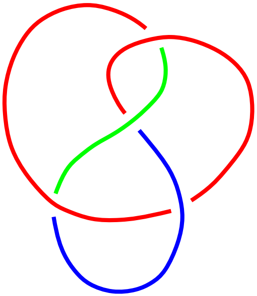

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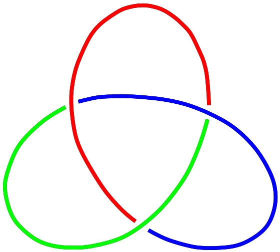

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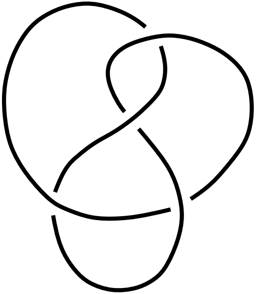

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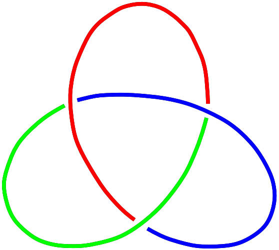

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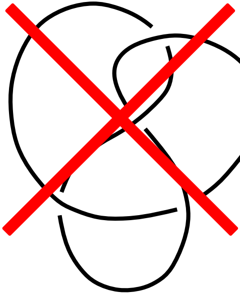

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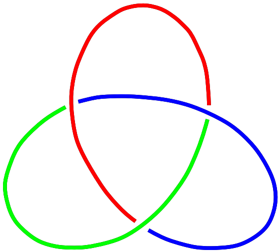

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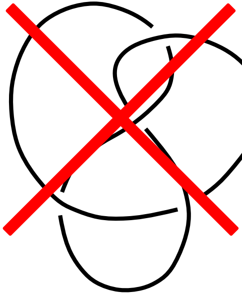

The figure-eight knot is not 3-colorable.

Does it Depend on the Diagram?

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Trefoil:

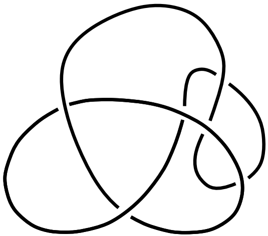
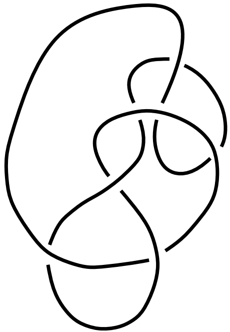


Figure-Eight:



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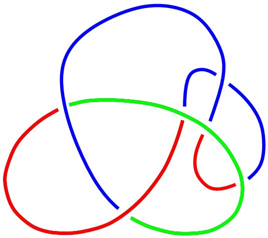
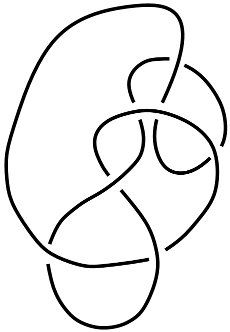


Figure-Eight:



Does it Depend on the Diagram?

Did we just make a lucky choice of diagram for the trefoil? Were we unlucky with our choice of diagram for the figure-eight knot?

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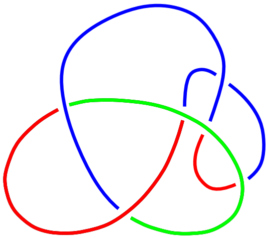

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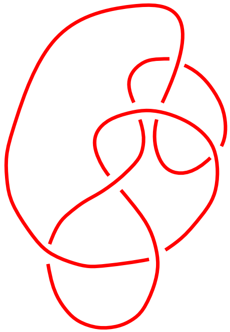

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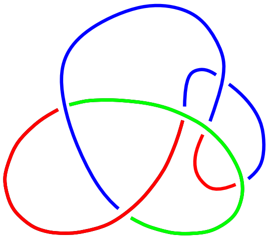

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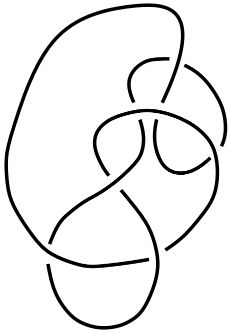

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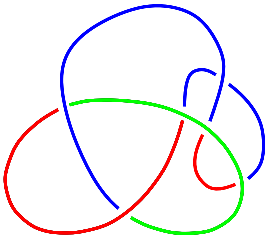

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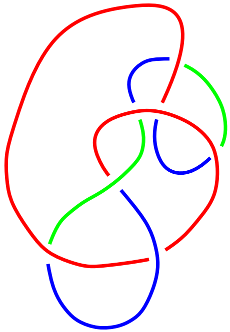

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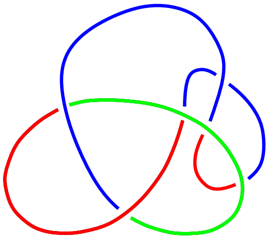

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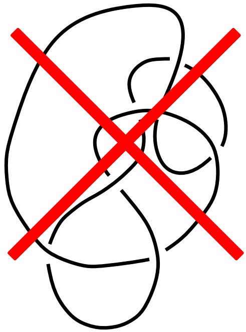
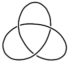
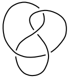


3-Colorability Theorem

Theorem

Let K be a knot. Then either every diagram of K can be 3-colored, or none of them can be 3-colored.

This tells us that 3-colorability only depends on the knot K, and not on the diagram we've drawn.

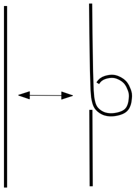
Thus the trefoil  and the figure-eight knot  are genuinely different knots (and not just different pictures of the same knot).

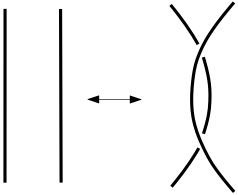
3-colorability is a knot invariant.

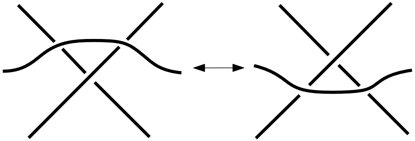
3-Coloring Proof

Proof.

Starting with a diagram of a knot, it can be transformed into any other diagram by a sequence of Reidemeister moves:







□

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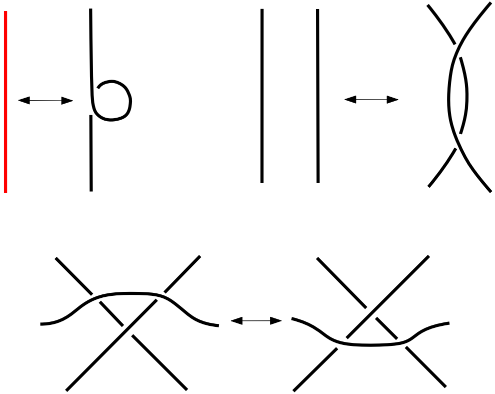

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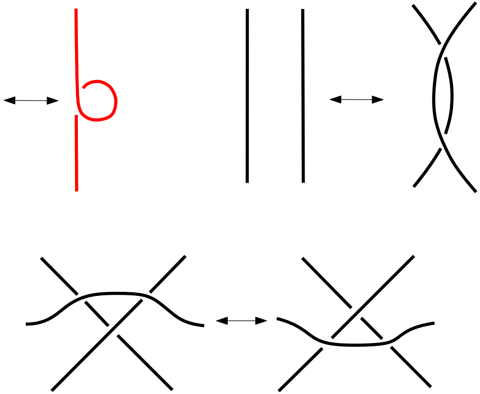

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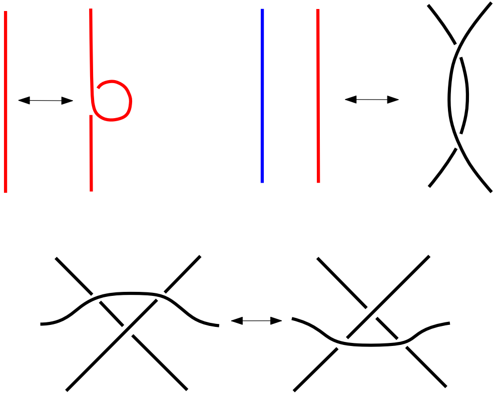

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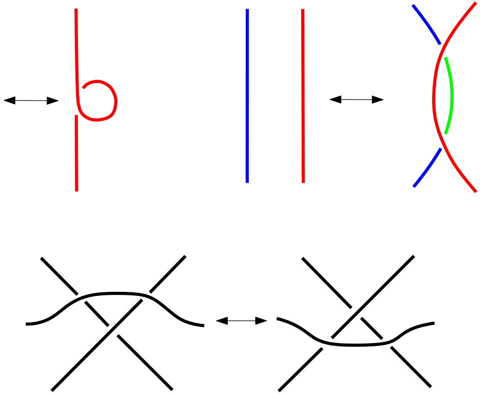

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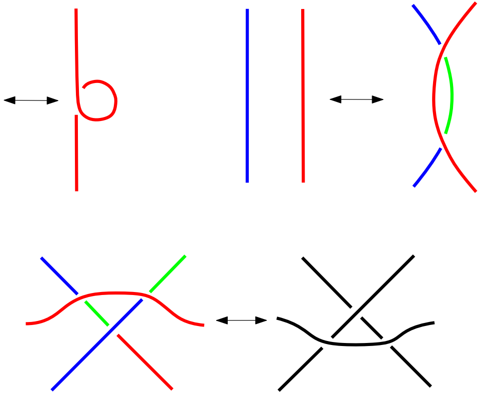

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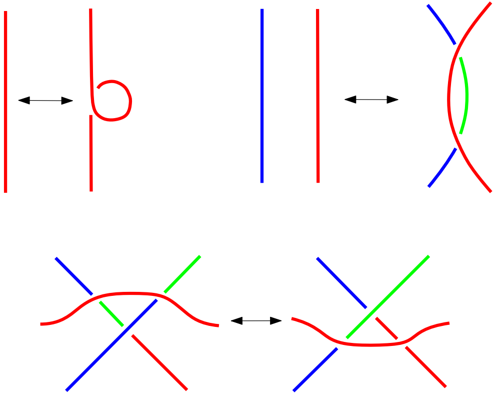

□

Other Knot Invariants

Knot invariants provide information about knots, and help distinguish different knots.

Invariants can be numbers, polynomials, or other algebraic objects.

Thank You!!!