MATH CIRCLE SESSION

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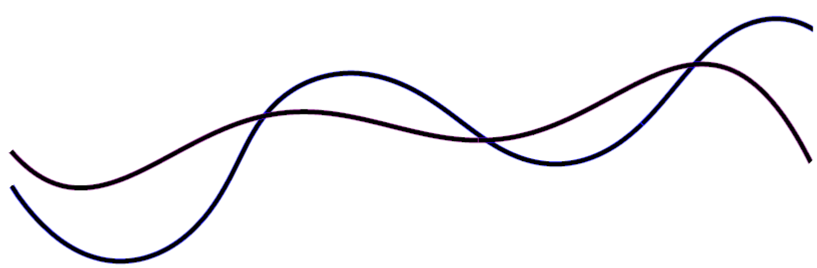
THE MATHEMATICS OF BICYCLE TRACKS

OVERVIEW:

You are brought to a crime scene. You are told that a thief just made off with a bag full of diamonds, escaping on a bicycle. You come across the following pair of bicycle tracks in the snow, no doubt made by the fleeing thief. But which way did the thief go?



Just by looking at the shapes of the tracks (tread marks, splashes of snow are inconclusive), can you determine which way the thieving cyclist went: left to right or right to left?

CONDUCTING THE ACTIVITY

Supplies Needed:

A bicycle
Sidewalk chalk
About 40 feet of poster paper
Thick red markers and thick blue markers
Painter's tape
Yard sticks

PART I: WHICH WAY DID THE BICYCLE GO?

Start by telling the story of the overview, sketching, very roughly, a pair of bicycle tracks on the board. Ask:

Does it make sense that there are two tracks?

Answer: A bicycle has two wheels and each wheel leaves a track.

Now mention that your sketch is highly inaccurate, that we actually need a genuine pair of bicycle tracks in order to properly observe their mathematical structure.

Ask participants to cover each wheel of a bicycle with chalk, one wheel with blue chalk say, and the other red chalk. Roll out the poster paper in a corridor and ask a volunteer to ride a wobbly path down the paper roll (making sure to not head off the paper, but also to give a path with lots of left and right turns).

The chalk marks on the paper will likely be faint. Ask participants to draw over the colored traces with markers of the same colors.

Now bring the paper back into the room and hang it on the wall with painter's tape. Ask folk to forget what they just witnessed and ask:

Okay. You just came across these very tracks. Is it possible to determine which way the bicycle went?

Two nudging questions:
Ask these questions only when/if frustration becomes too much.

> *Is it possible to say which track is most likely to be the back wheel track and which is the front wheel track?*
>
> **Answer:** The more "stable" track is likely to be the back wheel track and the wobblier one the front wheel track.

Later…

> *You said that the back wheel is more "stable." In what sense? What makes it stable? What do you notice about the construction of a bicycle?*

Hold up a bicycle. Observe that the back wheel is fixed in its frame and always points towards the point of contact of the front wheel to the ground - and a fixed distance to boot (the distance between the axles of the wheels).

This observation gives the puzzle away.

The solution:
Participants will likely state something loose along the lines:

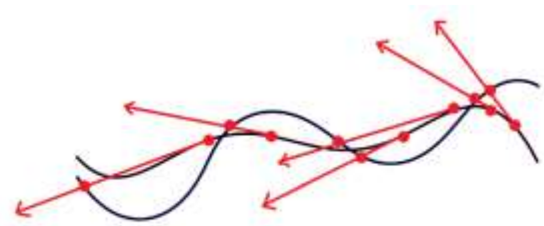
> "So each point on the back wheel track should point towards the front wheel track at a fixed distance."

Help guide a more precise description, something along the lines:

> The tangent line at each point on the back wheel track intercepts the front wheel track at a fixed distance.

Hold up a yard stick as a tangent line at a point on the back wheel track. Demonstrate that, in moving the along the back track, the tangent line does indeed intercept the front wheel track at some consistent fixed distance in one direction of motion but not in the other.

Left to Right ✓



Right to Left ✗

We have, it seems:

For a given pair of bicycle tracks, it is not only possible to determine in which direction the bicycle traveled, but also the length of the bicycle that made those tracks.

TOUGH QUESTIONS:

Mention that there is at least one instance in which we cannot determine the direction of motion of the cyclist:

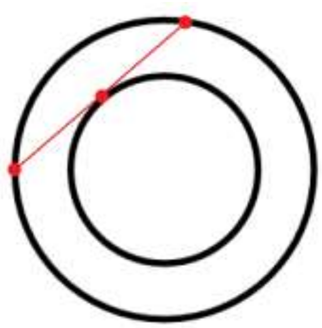
If the rider rides a perfectly straight path, the two tracks of the wheels overlap along a single straight line. It is impossible in this case to ascertain the direction of motion from analyzing the tracks.

Question: *Is this the only example of rider motion for which we could not determine the direction of travel?*

Answer: No. Suppose a cyclists rides in a perfect circle.

(It is worth pausing here to ask: In this case, what do the two tracks look like? Can you still determine which track is the back wheel track and which comes from the front wheel?)

Because of the symmetry of the situation we cannot determine the direction of motion in this case too.



SURPRISINGLY TOUGH QUESTION: Are these the only two cases in which we could not determine the direction of travel?

More precisely:

1. *If a pair of smooth curves have the property that each tangent line to one curve always intercepts the other curve at two locations, a fixed distance* r *either side of the point of contact along the tangent line, must the curves each have constant curvature?*

TWO OTHER REALLY TOUGH QUESTIONS:

2. *Could two bicycles of different lengths produce the same pair of (non-straight) bicycle tracks?*

3. *Could a bicycle produce a single non-straight track? (That is, is it possible to ride a non-straight path on a bicycle so that back wheel track covers the front wheel track? Alternatively, could a bicycle and a unicycle produce exactly the same non-straight track?)*

Comment: All three questions can be united as one: *Could two bicycles of lengths r and s produce the same non-straight tracks?* (Allow the cases $s = -r$ and $s = 0$ too.)

These questions are fun to explore, but they will require some technology help in order to draw bicycle tracks with the correct mathematical structure. (Write a set of parametric equations for a back wheel curve, compute the unit tangent vector at a location on this curve and plot the endpoint of this vector. The trace of this point is the front wheel track.)

Mention these tough questions, but perhaps move on to part 2 of this activity after just a short discussion and period of mulling.

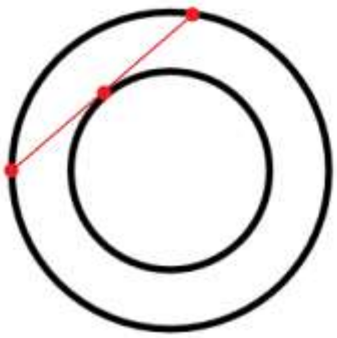
Reference: I first learned of this problems in Joseph Konhauser, Dan Velleman, and Stan Wagon's text *Which Way did the Bicycle Go? … and Other Intriguing Mathematical Mysteries* (MAA, 1996). They learned of the problem from a geometry course being developed at Princeton in the 1980s.

This problem is described in Sherlock Holmes novel, but Doyle, according to the three authors above, gives an incorrect (non-mathematical) solution for determining the direction of travel.

A solution to Problem 3 appears in Dr. David Finn's 2002 paper "Can a Bicycle Create a Unicycle Track?" https://www.maa.org/sites/default/files/pdf/upload_library/22/Polya/Finn.pdf. This, and question 1, is also discussed in Stan Wagon's book *Mathematica In Action: Problem Solving Through Visualization and Computation* (Springer).

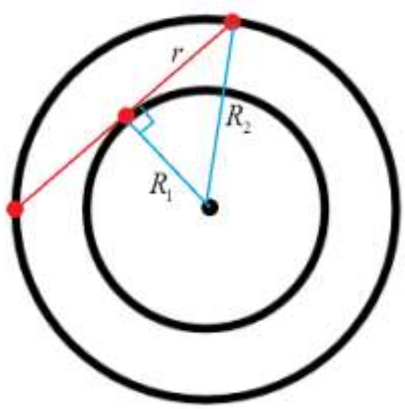
PART 2: THE AREAS BETWEEN TRACKS

Back to the pair of circular tracks.



Question: *If the distance between the two wheels of the bicycle that made this pair of tracks is* r *units, what is the area between the two tracks?*

Answer: Surprisingly the answer is πr^2 no matter the sizes of the concentric circles.



Let R_1 and R_2 be, respectively, the inner and outer radii of the concentric circles.

A tangent line to a circle is perpendicular to the radius of the circle at the point of contact. In our diagram we thus see a right triangle and the Pythagorean Theorem gives $R_1^2 + r^2 = R_2^2$.

The area between the two circles, that is, between the two tracks, is:

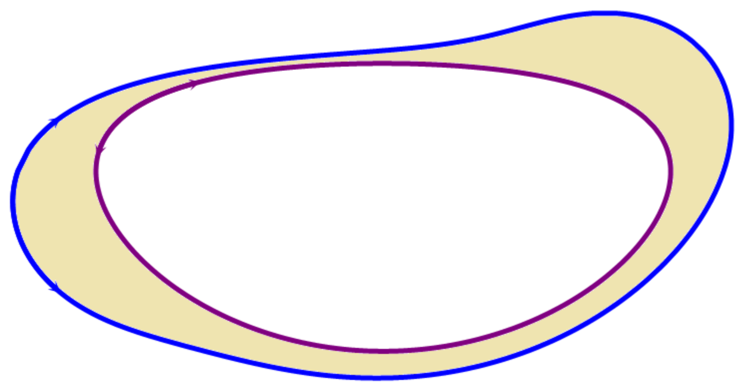
$$\begin{aligned} \text{area} &= \pi R_2^2 - \pi R_1^2 \\ &= \pi\left(R_2^2 - R_1^2\right) \\ &= \pi r^2. \end{aligned}$$

A BOLD CLAIM:

Draw a rough diagram on the board of a pair of tracks that result from riding in a non-circular convex loop.

Here is a bold claim:

The area between these tracks is again πr^2*. In fact, this is the case for any pair of tracks that result from riding the bicycle in some (convex) loop.*



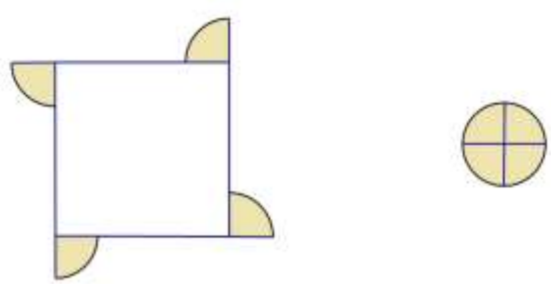
Could this claim possibly be true?

(By the way, the above pair of curves pictured were generated with computer software and are mathematically correct tracks. Can you see which way the bicycle traveled?)

Towards the claim:

Appropriately lead participants to the following ideas.

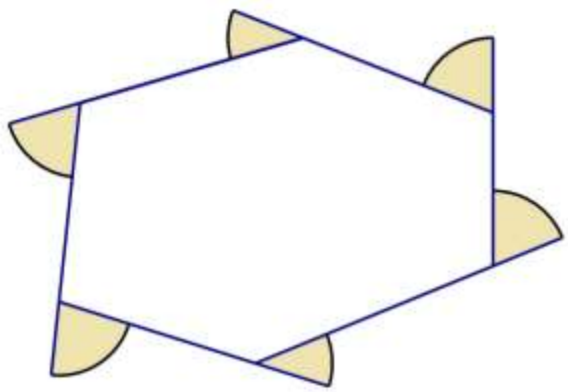
Suppose we force the back wheel of the bicycle to follow a square path, pivoting the front wheel about its corners. Can you see what with the front wheel path will look like? Can you see that the area between these tracks is certainly πr^2 ?

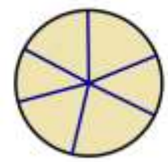


Can you see this is the case too for a rectangular back wheel track?

Explore triangular back-wheel paths. Other quadrilateral paths Other polygonal paths.

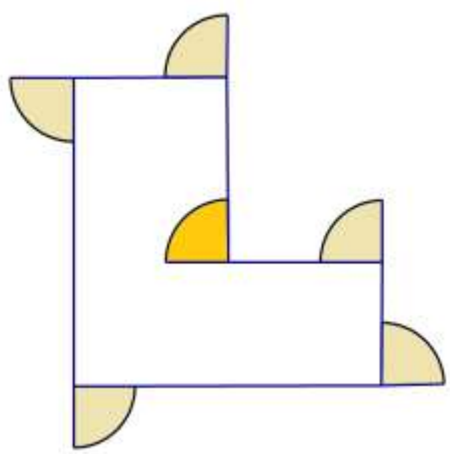
Answers: If we force the back wheel to follow any convex polygonal path, the region between the front wheel and back wheel tracks is composed of a set of sectors. Because the exterior angles of a regular polygon sum to 360°, these sectors fit together perfectly to make one full circle.





Thus the area between the front-wheel and back-wheel tracks for a cyclist riding in a convex polygonal path is πr^2.

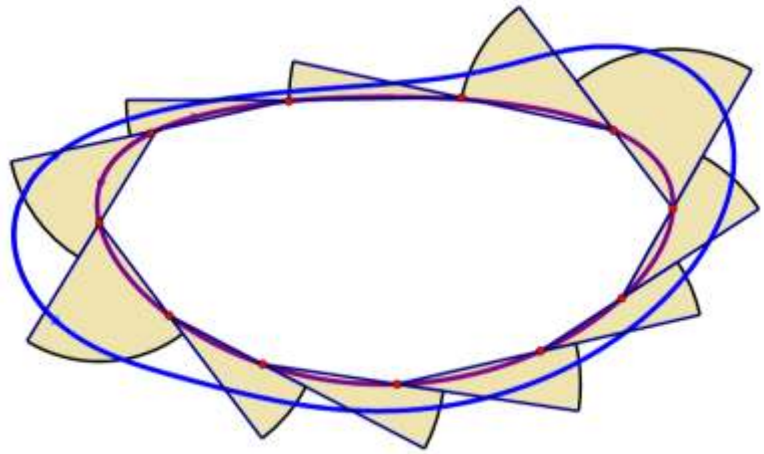
Side Exploration: Is the same true for a cyclist riding a non-convex polygonal paths?



Answer: Yes – if you regard areas swept out in a counter-clock and clockwise directions as having opposite signs!

Explaining the Bold Claim:

Any smooth convex curve can be approximated as a convex polygon by drawing short line segments between points of the curve. And if we ride the bicycle along the convex polygonal approximation we know the area between the bicycle tracks approximating the original tracks is sure to be πr^2. This approximates the area between the original pair of tracks.



We can get a better approximation of the true area between the tracks by using a polygon to approximate the back wheel track with many more sides composed of much shorter line segments. But we know that area between the tracks in this approximation will be πr^2 too. We can do finer and finer approximations, but each time these approximations have value πr^2. Since the approximate values converge to the true value of the area between the original tracks, we can only conclude that the area between the original tracks must be πr^2 too.

Comments:

If one rides a non-convex curve, then the area between the tracks is again πr^2, as long as one counts are swept out in opposite directions (clockwise or counter-clockwise) as opposite in sign.

In a convex loop one undergoes, overall, one full turn. The overall effect of riding a figure eight, for example, is no turning. The area between the tracks in this case (counted with sign) is 0. If one rides a loop that, in overall effect, undergoes n full turns, then the area swept out between the tracks "accumulates" to the value $n\pi r^2$.

COMMON CORE CONNECTIONS:

Mathematical Practice Standards
Every math circle activity, by very definition, engages participants in the first, and most important, Mathematical Practice Standard:

> **MP1:** *Make sense of problems and persevere in solving them.*

And, because of the collaborative nature of math circles, the third practice standard is in the fore as well:

> **MP3:** *Construct viable arguments and critique the reasoning of others.*

In these two bicycle-track problems, one must use precise language to communicate key ideas (we must speak of “tangent lines” to communicate “the direction the back wheel points,” for example), which is the spirit of:

> **MP6:** *Attend to precision.*

One can even argue, especially with the open problems in part 2 of this activity, we must also attend to:

> **MP5:** *Use appropriate tools strategically.*

Content Standards
In terms of the content standards this activity connects only tangentially.

The concept of tangent lines are only brought up in the context of circle geometry (high school, G.C.2 and G.C.4) and the sum of the exterior angles to polygons are mentioned only with regard to triangles (grade 8, 8.G.5).

One doesn’t actually need to know that formula for the area of a circle for this activity (only that the sectors we create in polygonal bicycle motion fit together to make a full solid circle), but this activity could be used to reinforce the content standards about this (grade 7, 7.G.4 and high school G-GMD.1).

We did use the fact at one point in this activity that a tangent line to a circle is perpendicular to the radius of the circle at the point of contact (high school G.C.2) and we made use of the Pythagorean Theorem (grade 8, 8.G.7).

> **7.G.4.** *Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.*

> **8.G.5.** *Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.*

8.G.7. *Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.*

G.C.2. *Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

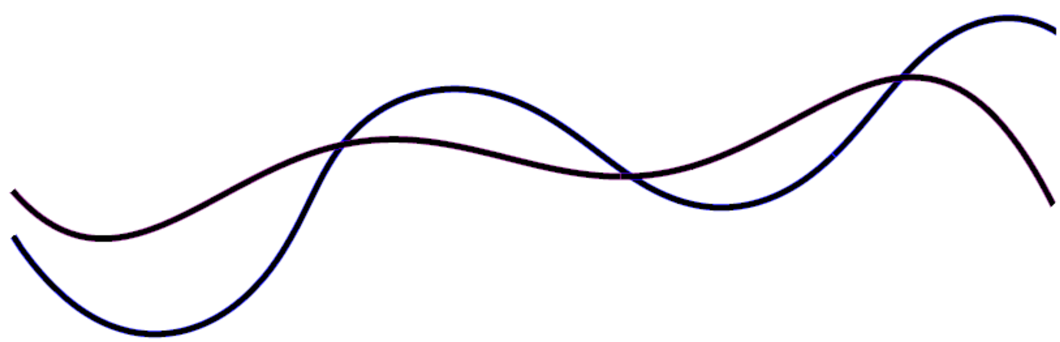
G.C.4. (+) *Construct a tangent line from a point outside a given circle to the circle.*

G-GMD.1. *Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

HANDOUTS:

WHICH WAY DID THE BICYCLE GO?

You are brought to a crime scene. You are told that a thief just made off with a bag full of diamonds, escaping on a bicycle. You come across the following pair of bicycle tracks in the snow, no doubt made by the fleeing thief. But which way did the thief go?



Just by looking at the shapes of the tracks (tread marks, splashes of snow are inconclusive), can you determine which way the thieving cyclist went: left to right or right to left?

WHICH WAY DID THE BICYCLE GO?