Quilting Squares in a Math Circle

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Joint work with Beth Malmskog

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VILLANOVA
UNIVERSITY

People and Places

Figure: Beth Malmskog, foreground. Wissahickon River Goddess, background (Swann Memorial Fountain, Philadelphia, PA).

PhD + Epsilon Blog (Editors - Beth Malmskog and Sara Malec): `http://blogs.ams.org/phdplus/2016/05/29/constructive-summer/`

People and Places

- Beth and I started the Graterford Math Circle in 2016.
- GMC serves incarcerated men who have been part of the Villanova degree program at SCI Graterford.
- Meetings are once a month, with 5 to 10 participants.

Figure: The State Correctional Institution at Graterford.

The Quilt Problem

Judy's question about round robin quilt exchanges...

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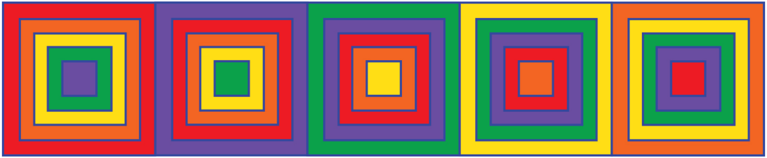

Figure: Five quilts created by passing in a cycle. This method was unsatisfactory because each quilter always passed to the same neighbor.

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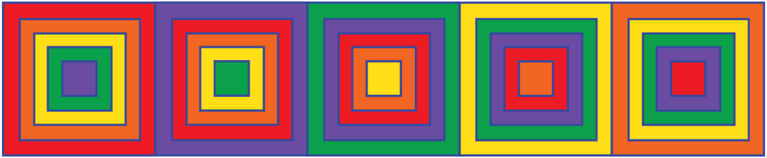

Figure: Five quilts created by passing in a cycle. This method was unsatisfactory because each quilter always passed to the same neighbor.

"I'd like to arrange the trades so no one ever gets their quilt from the same person, but I can't quite figure it out. I tried it with 5 people in a group and, while everyone gets the quilt to sew on, the quilts are passed twice to the same person.
I think we will pretty much always have 5 people in a group, but it is possible that we would have groups with 4 or 6 people. Is there something that will work for that, too?"

Rules recap

- Each person has a quilt at each stage of the exchange
- Once person A passes a quilt to person B, the pass from A to B will never happen again (but the pass from B to A can still occur)
- Questions about how it works?

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Question (3.)

Can you solve Judy's problem of finding a quilt passing pattern for 5 people? If yes, demonstrate it. If not, justify why not.

Question (4.)

Can you describe a general method of passing quilts for an even number of people?

Spoiler Alert!!

No round robin quilt exchange (following the rules) exists for 3 or 5 people.
For 4 people, the following will work:

Figure: A Meta-Quilt showing a passing pattern for four quilters, along with the resulting concentric quilts.

Another Visual

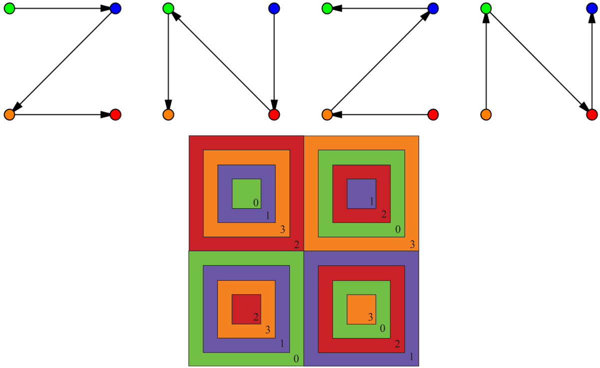

Figure: Another way to imagine the passing pattern for four quilters, along with the resulting quilts.

Neat Math Connections

- Directed graphs
- Row complete latin squares
- Sequenceable groups

Connections: Latin Squares

Definition

Let S be a set of n distinct elements. An $n \times n$ array filled with elements from S such that each element appears once in each row and once in each column of the array is called a *latin square* of order n.

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Example: Sudoku requires a latin square of order 9.

8	9	7	1	6	2	4	5	3
5	1	3	8	7	4	9	6	2
2	4	6	9	3	5	1	8	7
3	6	9	7	8	1	5	2	4
7	5	2	4	9	6	3	1	8
1	8	4	5	2	3	7	9	6
6	7	5	3	1	8	2	4	9
4	3	8	2	5	9	6	7	1
9	2	1	6	4	7	8	3	5

Figure: A solved Sudoku puzzle.

Quilt exchange $\leftrightarrow$ latin square

The 5 quilts created by a cycle of passing can be described by a latin square.

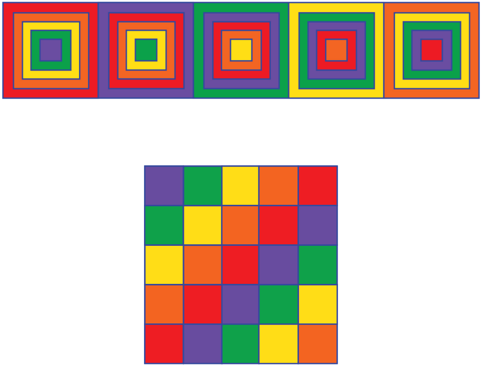

Figure: The latin square of order 5 associated to the cyclic quilts.

Qualities of a ‘good’ latin square

- Think of each row as the path of a quilt through the quilters, labeled 0 through $n-1$.
- The condition that no quilt undergoes the same pass twice means that in the latin square, the sequence (i, j) appears at most once in the rows of the square.
- There are $n(n-1)$ different sequences (i, j), and $n(n-1)$ quilt passes must take place, so in fact each sequence (i, j) must appear exactly once in the array.

RCLS(n) and Williams' construction

Definition

An $n \times n$ latin square with the special quilt property is called a *row complete latin square* of order n, abbreviated RCLS(n).

RCLS's were first studied by E. J. Williams in 1948 to develop designs for experiments in which treatments might have residual effects, i.e., the milk production of dairy cows on various feeds.
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Example

0	1	5	2	4	3
1	2	0	3	5	4
2	3	1	4	0	5
3	4	2	5	1	0
4	5	3	0	2	1
5	0	4	1	3	2

Table: An example of Williams' construction of an RCLS(6).

Quilts on graphs

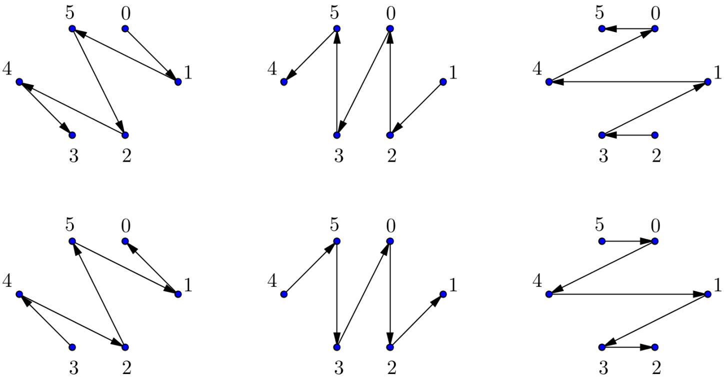

Figure: Graphs describing the paths of the six quilts in the previous example.

This method works for any even n.

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- By Williams' construction, an RCLS(n) always exists when n is even, so these exchanges can always be arranged for even-sized quilting circles.
- **Extension:** Can we completely classify the possible sizes of quilting circles for which these exchanges are possible—that is, can we determine the values of n for which an RCLS(n) exists?

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- It is conjectured that there are no row complete latin squares of prime order, but proving this is still open.

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We're happy to share materials!!! Email:
`kathryn.haymaker@villanova.edu`