Making Infinitely Many Mistakes Deliberately – Iteration

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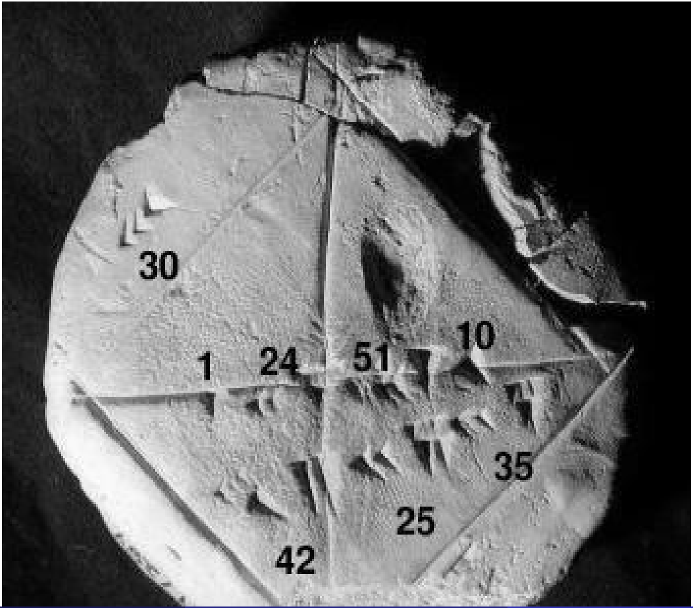
- I allowed some time for speculation on the square root of 2 and how we could discover that fact even without the Pythagorean theorem. Then we speculated on the calculation.
- For kids, the idea of deliberately solving the wrong equation infinitely many times was novel and amusing, so we had sustained interest.

YBC 7289

YBC 7289 with some annotation



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- Since many student know some decimals expansions for $\sqrt{2}$, we can convert this to decimal if requested.
- Using Mathematica, we find the number on the tablet middle line is the (rounded) 8 place decimal 1.4142130, while the approximate value from Mathematica for $\sqrt{2}$ is 1.4142136

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- Most students can find an argument for this eventually.
- One possible side branch to the discussion here is to see if students can generalize the argument to one for a general rectangle, i.e. the Pythagorean Theorem. If it comes up entirely from students, I would likely follow that lead but if not, I would stick to the main story line, but you could choose otherwise.

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- We are aiming for an iteration that creates better approximations. There is an algebraic route and a geometric route for us (it is not altogether clear how the Babylonians did it as far as I know).

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- This is now a new rectangle. Is it a square? Why or why not?
- It turns out to be a non-square, but “more square-ish”. We made an approximation and now we can iterate it. The new sides are now inequalities bracketing $\sqrt{2}$.

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- We again make a deliberate mistake and drop r_n^2 to find x_{n+1}. The new sides are now inequalities bracketing $\sqrt{2}$. If we make infinitely many mistakes deliberately we can in principle compute $\sqrt{2}$ exactly.

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- The harmonic mean is in the picture also.
- The sequence is rational but $\sqrt{2}$ is never rational. The sequence never gives us a square, but limits do. There is a Diophantine equation lurking.

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- Solving by iteration in other contexts is fun too ... I really like doing the geometric series as an iteration problem.
- Solving linear systems by iteration is also a nice extension.
- And yet the numbers we generated are not quite the Babylonian number that is given (which seems to be linked to needing approximations for reciprocals also).

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- **THANK YOU!!**