

MATH WRANGLE 2016

SOLUTIONS

- 1 $P(2016)$ can equal any real number *except* 2016. To see why, define the polynomial $Q(x) = P(x) - x$. Clearly $Q(x)$ is of degree 2015, and

$$Q(1) = Q(2) = \cdots = Q(2015) = 0.$$

Since we know the zeros of Q , we can factor it and we have

$$Q(x) = C(x-1)(x-2)(x-3)\cdots(x-2015),$$

where C can be any *non-zero* constant. Hence $Q(2016) = 2015!C \neq 0$, which implies that $P(2016) \neq 2016$. However, we can assign any other real value to $P(2016)$ with an appropriate choice of C . For example, if $C = \frac{\pi - 2016}{2015!}$, then $Q(2016) = \pi - 2016$, and $P(2016) = \pi$.

- 2 **It is possible, with $N = \frac{2016}{48} \cdot 8 = 336$, with the a_k repeating the 8-term pattern $-1, 1, 1, -1, 1, -1, -1, 1$.** Use the observation that if $f(x)$ is a degree- k polynomial, then for any constant h , the difference $f(x+h) - f(x)$ will be a degree- $(k-1)$ polynomial. If we iterate this process three times, we can find a way to manipulate consecutive cubes to always get a *constant* which divides 2016.

More precisely, let $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7 \dots$ be consecutive cubes. In other words, $c_m = (m+u)^3$ for some fixed starting integer u . Then the differences $a_i := c_{i+1} - c_i$ will be quadratic functions of u (depending on the parameter m , as well). Continuing, we see that the differences $b_j := a_{j+2} - a_j$ will be linear in u (again depending on m) and thus finally that the quantity $b_4 - b_0$ is independent of m and u . Thus we have that

$$b_4 - b_0 = a_6 - a_4 - (a_2 - a_0) = a_6 - a_4 - a_2 + a_0 = c_7 - c_6 - c_5 + c_4 - c_3 + c_2 + c_1 - c_0$$

is constant and in fact equals 48. In other words, we can conclude that for any value of u

$$s_u := -u^3 + (u+1)^3 + (u+2)^3 - (u+3)^3 + (u+4)^3 - (u+5)^3 - (u+6)^3 + (u+7)^3 = 48$$

Clearly then we can use this method to produce a "sum-difference" of any value that is a multiple of 48, and of course $2016 = 48 \cdot 42$ fits this requirement.

- 3 The answer is $16/21$. First of all, the probability that Anna wins a round of the game is $2/3$. This can be determined by computing an infinite series or by noticing that if x is the probability of winning a round, we have $x = 1/2 + x/4$, since either Emilee wins on the first flip (with probability $1/2$) or she flips tails and Zach flips tails (with probability $1/4$)

and now the game is reset and she has a probability of x of winning. Solving, we find that $x = 2/3$.

Here is a reformulation of the problem. Start at the value 2 on the number line, and consider a random walk where there is a $2/3$ probability of moving one unit to the right and $1/3$ probability of moving one unit to the left; we want to find the probability that we reach 6 before we reach 0.

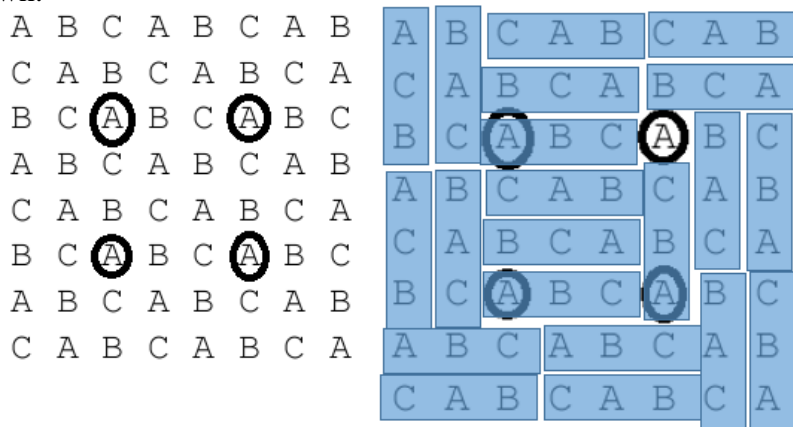
Now define a_k to be the probability of reaching 6 before reaching 0 when starting from k . Clearly $a_0 = 0, a_6 = 1$ and we wish to compute a_2 .

We have the following equations:

$$\begin{array}{rcl}
 a_1 & = & \frac{1}{3}a_0 + \frac{2}{3}a_2 = \frac{2}{3}a_2 \\
 a_2 & = & \frac{1}{3}a_1 + \frac{2}{3}a_3 \\
 a_3 & = & \frac{1}{3}a_2 + \frac{2}{3}a_4 \\
 a_4 & = & \frac{1}{3}a_3 + \frac{2}{3}a_5 \\
 a_5 & = & \frac{1}{3}a_4 + \frac{2}{3}a_6 = \frac{1}{3}a_4 + \frac{2}{3}
 \end{array}
 \quad \Rightarrow \quad
 \begin{array}{rcl}
 3a_1 & = & 2a_2 \\
 3a_2 & = & a_1 + 2a_3 \\
 3a_3 & = & a_2 + 2a_4 \\
 3a_4 & = & a_3 + 2a_5 \\
 3a_5 & = & a_4 + 2
 \end{array}
 \quad \Rightarrow \quad
 \begin{array}{rcl}
 a_2 & = & \frac{3}{2}a_1 \\
 a_3 & = & \frac{7}{4}a_1 \\
 a_4 & = & \frac{15}{8}a_1 \\
 a_5 & = & \frac{31}{16}a_1 \\
 2 & = & \frac{63}{16}a_1
 \end{array}$$

So at last we have $a_1 = \frac{32}{63}$ and thus $a_2 = \frac{16}{21}$.

- 4 Surprisingly, one can only tile the chessboard-minus-one-square for just four choices of square to remove. To see this, color the chessboard with three colors, A, B, C , alternating colors as shown.



Notice that any tromino placed on this board will use exactly one square of each color. Also notice that this coloring scheme uses 22 As, 21 Bs, and 21 Cs. Consequently, if the board can be tiled, then the missing square must be colored A (so that the remaining 63 squares have exactly 21 of each color and are tiled by 21 trominos).

But just because we must remove an A-colored square doesn't mean that a tiling is possible. For example, suppose we removed the upper-left corner square, which is colored A. That certainly leaves 21 of each color, but it cannot be tiled: Suppose that it could be tiled. Then certainly, if we rotated the figure clockwise by 90 degrees, we would still have a tiling. But now the missing square is B, an impossibility!

Thus, the only A -squares that one can hope to remove and be able to tile the 63-square figure are those squares that are also colored A upon rotating by 90 degrees. There are four such squares which we have circled. The right side of the diagram shows a possible tiling when one of these is removed.

- 5 The minimum IQR is 10. We claim the following change gives the minimum: Change 17 and 21 to 43 and 44 on the lower half and change 66 and 75 to 46 and 46 on the upper half. This is a compensating change of 49 on both halves so the mean is unchanged and we have that

$$12 \ 33 \ Q_1 = \left(\frac{33+43}{2} \right) = 38 \ 43 \ 44 \ Q_2 = 46 \ 46 \ 46 \ Q_3 = \left(\frac{46+50}{2} \right) = 48 \ 50 \ 94$$

has an Interquartile range of $IQR = Q_3 - Q_1 = 48 - 38 = 10$.

To see that this gives a minimum IQR we note the following:

Since the median cannot change the net number of values changing sides with 46 must be zero. Since the mean cannot change all values which move up a total of x points must be matched by all the values which move down by a total of x points. Therefore, the desired changes must be produced by moving two values up in the bottom half (to at most 46) and then moving two values in the upper half down by the same amount to at least 46.

The following tables are generated by looking solely at one side or the other of the median/mean.

MOVE	DOWN	NEW Q_3	COMMENT
50 & 66 to 46 & 46	24	60.5	FORCES $IQR \geq 60.5 - 39.5 = 21$
50 & 75 to 46 & 46	33	51	FORCES $IQR \geq 51 - 39.5 = 11.5$
50 & 94 to 46 & 46	52	51	FORCES $IQR \geq 51 - 39.5 = 11.5$
66 & 75 to 46 & 46	49	48	ONLY VIABLE TOP SCENARIO
66 & 94 to 46 & 46	68	48	63 MAX ON LOWER HALF
75 & 94 to 46 & 46	77	48	63 MAX ON LOWER HALF

MOVE	UP	NEW Q_1	COMMENT
12 & 17 to 46 & 46	63	39.5	BOTTOM SCENARIO 1
12 & 21 to 46 & 46	59	39.5	BOTTOM SCENARIO 2
12 & 33 to 46 & 46	47	33.5	FORCES $IQR \geq 48 - 33.5 = 14.5$
17 & 21 to 46 & 46	54	39.5	BOTTOM SCENARIO 3
17 & 33 to 46 & 46	42	33.5	FORCES $IQR \geq 48 - 33.5 = 14.5$
21 & 33 to 46 & 38	38	31.5	FORCES $IQR > 48 - 31.5 = 16.5$

Since up/down values don't match between the top and bottom scenarios there must be a compromise. The up/down values in the table represent maximum possible changes so

no compromise can move more than these and this why we must choose the 49 in our first table (row 4) and a portion of bottom table scenario 3. The above tables also explain why we can't move 1 value on one side and three on the other since one value on the top would make Q_3 at least $\frac{50+66}{2} = 58$ so the IQR would be greater than $58 - 46 = 12$ and one value on the bottom would make Q_1 at most $\frac{21+33}{2} = 27$ so the IQR would be greater than $46 - 27 = 19$.

6 The answer is 80. Let n be the number of steps. In 20 seconds, Anna walks up 20 steps and gets to the top. Consequently the escalator moves at a speed of $n - 20$ steps in 20 seconds. Likewise, it moves $n - 32$ steps in 16 seconds. Equating these two, we have $\frac{n-20}{20} = \frac{n-32}{16}$.

7 The answer is **1.57**.

With $u = \frac{1}{2}$ the sequence of terms can be described as follows:

For the numerator we let $b_1 = \sqrt{\frac{1}{2}}$ and then for $n > 1$ we have $b_n := \sqrt{\frac{1+b_{n-1}}{2}}$ and we define $a_n = \frac{\sqrt{\frac{1-b_{n-1}}{2}}}{2^n}$. Given that we are taking nested square roots of $1/2$, it is natural to think about iterating the sine and cosine half-angle formulas, (i.e. $\cos(\frac{\theta}{2}) = \sqrt{\frac{1+\cos\theta}{2}}$ and $\sin(\frac{\theta}{2}) = \sqrt{\frac{1-\cos\theta}{2}}$) starting with the angle $\pi/4$. Indeed we have that $b_n = \cos(\frac{\pi}{2^n})$ and $a_n = \frac{\sin(\frac{\pi}{2^n})}{2^{-n}} = 2^n \sin(\frac{\pi}{2^n}) = \frac{2^{n+1} \sin(\frac{2\pi}{2^{n+1}})}{2}$.

Now consider a regular k -gon inscribed in a circle of radius one. The area of such a shape is k -times the area of an isosceles triangle with equal sides of length 1 and vertex angle $\frac{2\pi}{k}$ (i.e. just $\frac{k \sin(\frac{2\pi}{k})}{2}$). Therefore we see that a_n is just half the area of a regular 2^{n+1} -gon inscribed inside a unit circle. When $n = 100$, this is a regular polygon with an area that is just an insignificant amount less than that of the area of the unit circle, which of course is π . Hence for large values of n , a_n is equal to $\pi/2$ minus a tiny error. To two decimal places, this is **1.57**, of course.

8 The answer is 8%. Suppose the legal speed is L mph, and let Sonia's speed be $L(1+s)$. Suppose there are c cars per mile. Then each hour, Sonia will pass scL cars going in her direction, because her relative speed is sL . Likewise, in an hour, she will see $L(2+s)c$ cars going in the opposite direction, since these cars are traveling at a speed of $2L + Ls$ relative to her. Solving $L(2+s)c : scL = 26 : 1$ yields $s = 2/25 = 8\%$.