"MATH TAUGHT THE RIGHT WAY":
CURRICULUM FROM OVERSEAS
ADAPTED TO U. S. EDUCATIONAL REALITY
AND IMPLEMENTED PARALLEL TO

by

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BERKELEY MATH CIRCLE
Director

Joint Mathematics Meetings

It's Circular: Conjecture, Compute, Iterate

Baltimore, January 2019

BERKELEY
MATH CIRCLE

- For about 500 elementary, middle & high school students

- Tuesdays, 5 - 8pm
- Wednesdays, 5 - 8pm
- University of California at Berkeley
- http://mathcircle.berkeley.edu

Welcome to the Berkeley Math Circle!

Donate to the BMC

The Berkeley Math Circle (BMC) is a weekly program for over 500 San Francisco Bay Area elementary, middle and high school students. The weekly sessions are held on Tuesdays evenings at the UC Berkeley campus. The program is jointly sponsored by the:

- UC Berkeley Mathematics Department
- UC Berkeley Statistics Department
- UC Berkeley Electrical Engineering and Computer Science (EECS) Department
- Mathematical Science and Research Institute (MSRI)
- Simons Institute for Theory of Computing
- Vanguard Charitable, and
- Parents' Contribution

Emulating famous Eastern European models, the program aims at drawing kids to mathematics, preparing them for mathematical contests, introducing them to the wonders of beautiful mathematical theories, and encouraging them to undertake future careers linked with mathematics, whether as mathematicians, mathematics educators, economists, or business entrepeneurs. Read more about the BMC.

Quiz:

1. The relation "is tangent to," defined on the set of circles in the plane is:
a) reflexive;
b) symmetric;
c) transitive?

2. Is reflection across a line in the plane a function or just a relation? How about
- translation in the plane,
- rotation in the plane, or
- measuring the area of polygons?

The Real Questions:

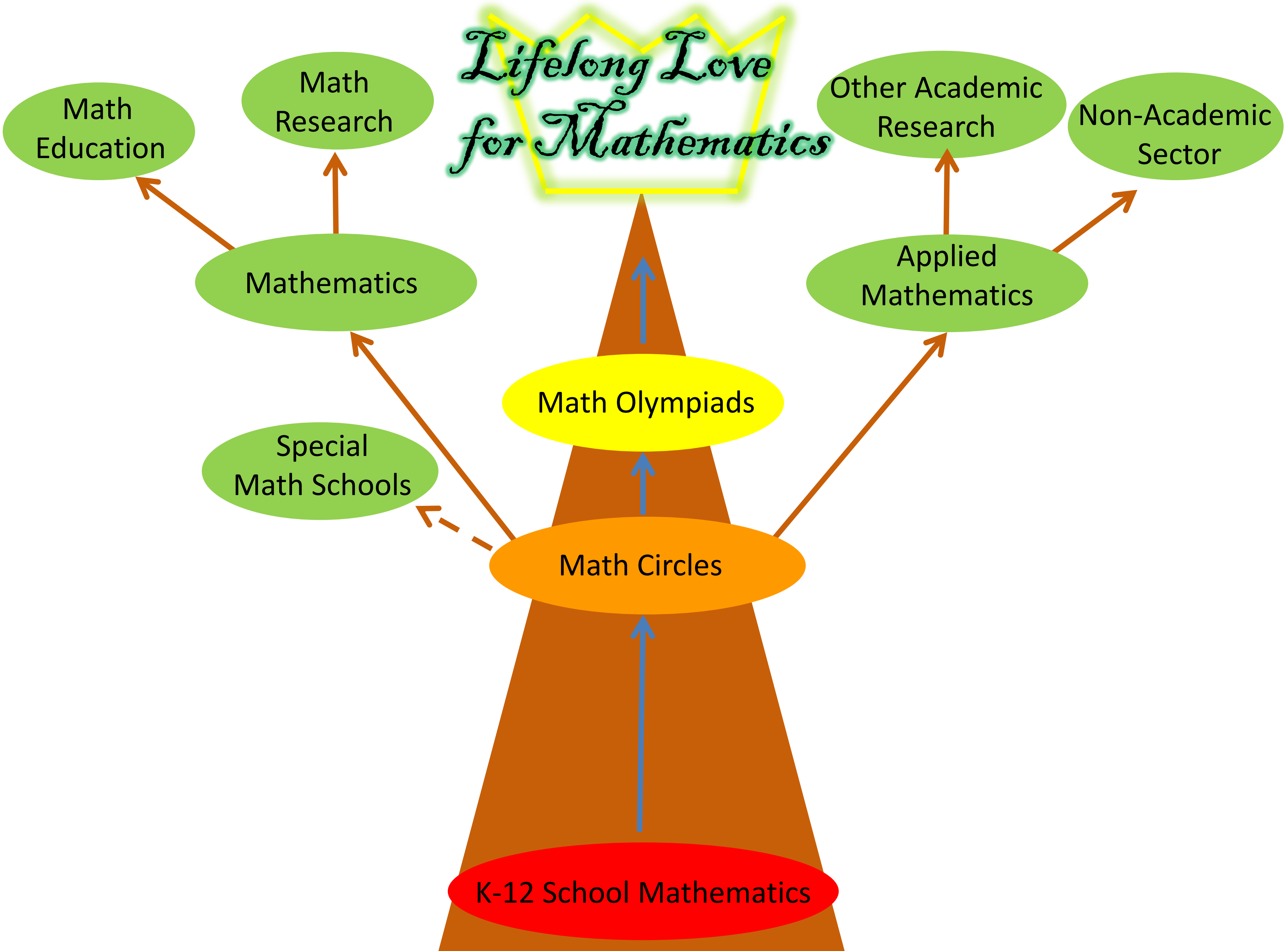
- For which grades/ages are these problem appropriate in the U.S.?
  Where do these problems fit within the US school curriculum? Do they fit at all?

- How many mathematical areas/subjects are incorporated in these questions?
  Should those areas be taught as separate subjects? Concurrently?

- Can/should teachers dare/dream of teaching these problems in U.S. school?

- In which grades in Eastern Europe were these problems been assigned to?
  In a regular school? Or in a special math-oriented school?

- What do these problems have to do with math circles and with this conference, in particular? Where do they fit in the overall discussion of math "outreach"?



Lifelong Love for Mathematics
Math Education
Math Research
Other Academic Research
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Mathematics
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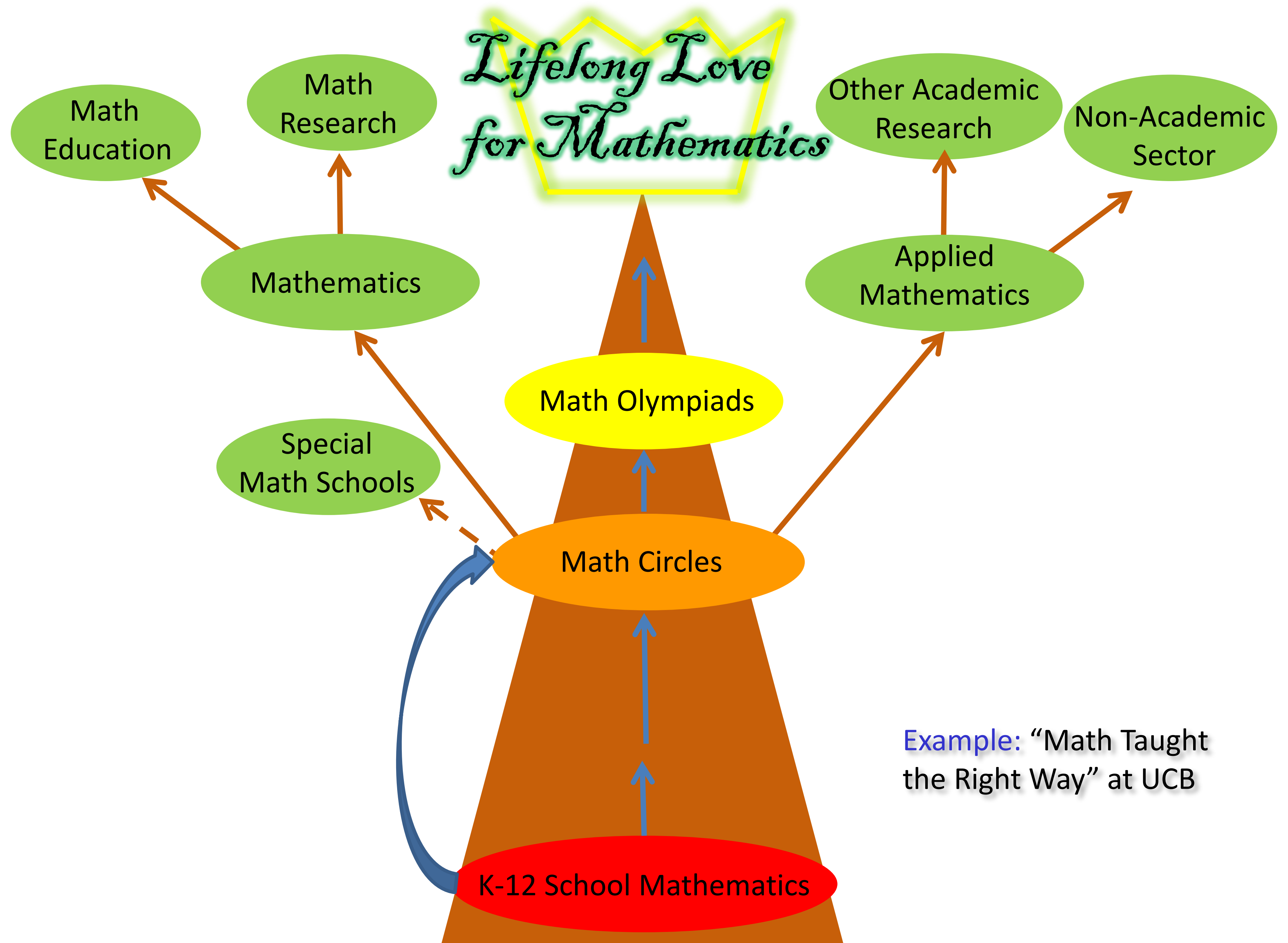
Math Olympiads
Special Math Schools
Math Circles
K-12 School Mathematics



Example: "Math Taught the Right Way" at UCB

Quiz:

1. The relation "is tangent to," defined on the set of circles in the plane is:
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All functions

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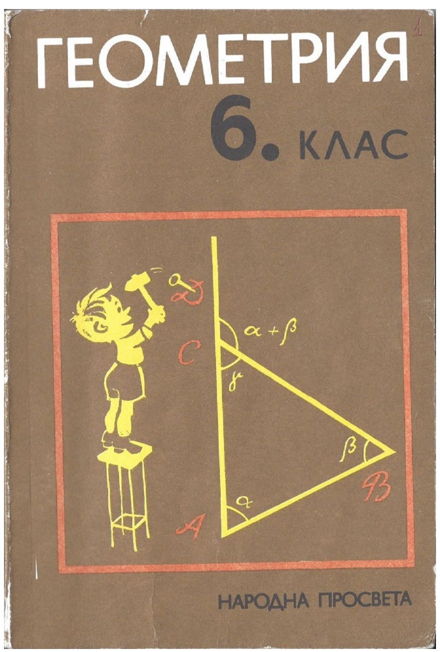
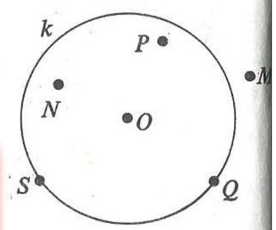
6



2. На черт. 62 са дадени окръжност $k(O;\ r=1{,}6$ см) и точките M, P, Q, S, D и N. Кои от тези точки са на разстояние от точката O:

а) не по-малко от 1,6 (см);
б) не по-голямо от 1,6 (см).



Черт. 62

3. Рефлексивна, симетрична или транзитивна е релацията . . . е допирателна на . . ., определена в множеството на окръжностите?

4. Ако k_1 и k_2 са две окръжности съответно с центрове O_1 и O_2 и радиуси r_1, r_2, като $r_1 > r_2$, вярно ли е, че:

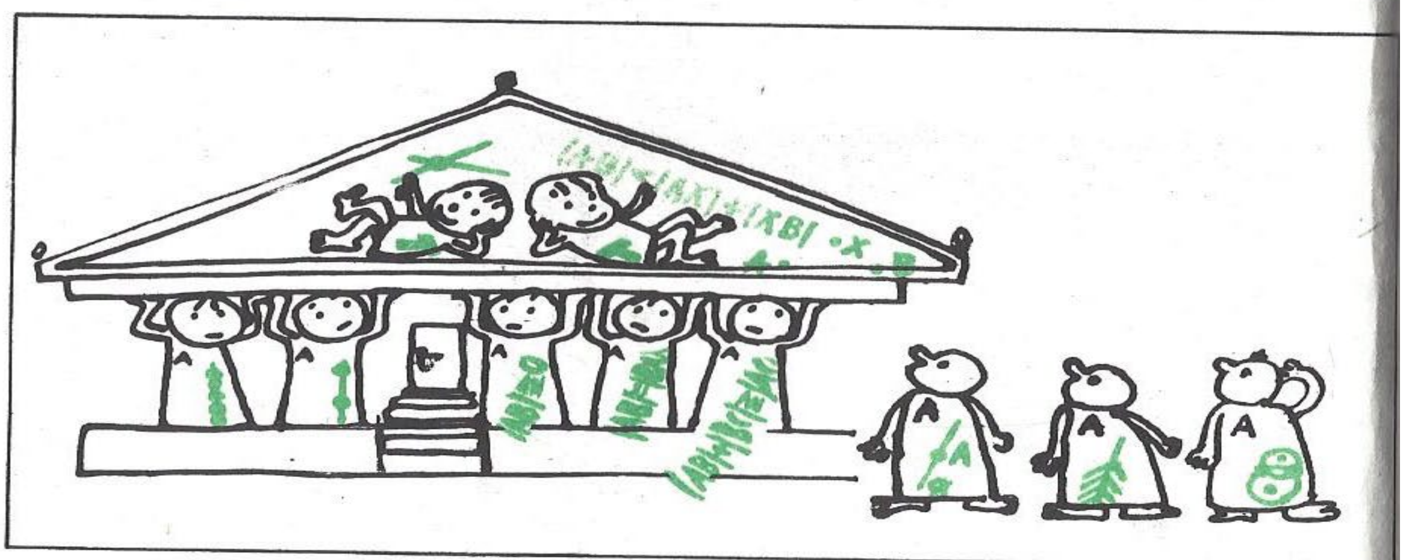
а) Ако $r_1 - r_2 \leqq O_1O_2 \leqq r_1 + r_2$, то $k_1 \cap k_2 = \varnothing$.
б) Ако $r_1 - r_2 \leqq O_1O_2 \leqq r_1 + r_2$, то $k_1 \cap k_2$ се състои от точно дв
точки.

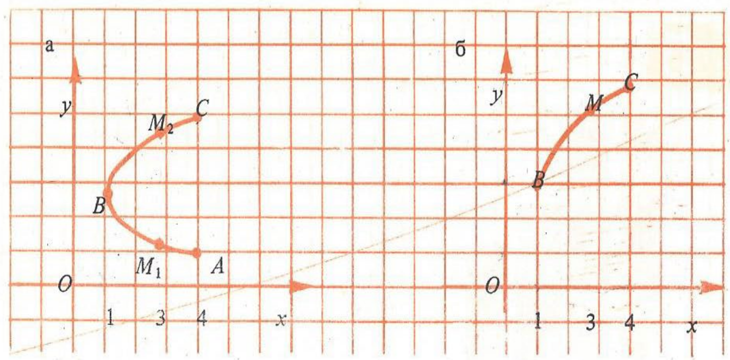
5. Рефлексивна, симетрична или транзитивна е релацият . . . е пресекателна на . . ., определена в множеството на окръж ностите.

6. Дадени са точка O_1 и окръжност $k_2(O_2;\ r_2=4$ см) така, ч $O_1O_2=6$ см. От колко точки се състои сечението на k_2 и окръж ността $k_1(O_1;\ r_1)$, ако:

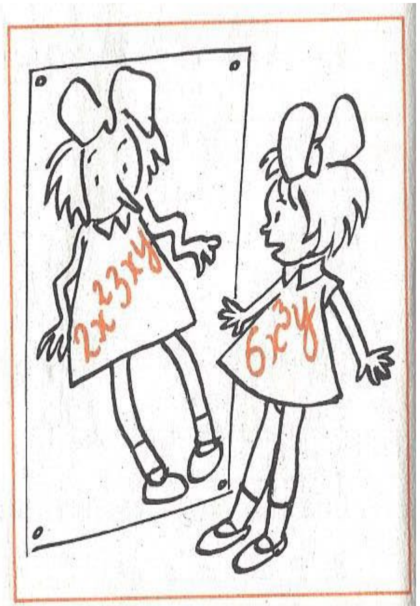
а) $r_1=2{,}4$ см;
б) $r_1=1{,}2$ см;
в) $r_1=2$ см.

7*. Постройте точка на разстояние a от точката A и н разстояние b от точката B при избрана мерна единица. Кои с условията от а) до г), ако са налице, такава точка сигурно съ шествува:





Фиг. 28



ЧАСТ ПЪРВА

РАЦИОНАЛНИ ЧИСЛА

(Преговор с допълнение)

1 ВИДОВЕ РАЦИОНАЛНИ ЧИСЛА. РАВЕНСТВО И НАРЕДБА НА РАЦИОНАЛНИТЕ ЧИСЛА

1. Видове рационални числа

Ние работим вече с рационални числа. *Множеството на рационалните числа се бележи с* **Q**.

Както знаем, *рационалните числа биват положителни, отрицателни или нула.* **Например** $+71$; $+\frac{3}{7}$; $+2{,}3$; ... са положителни (обикновено те се пишат без знак: 71; $\frac{3}{7}$; $2{,}3$), а -123; $-5{,}3$; $-3\frac{1}{2}$ са отрицателни числа.

Единствено числото 0 нито е положително, нито е отрицателно.

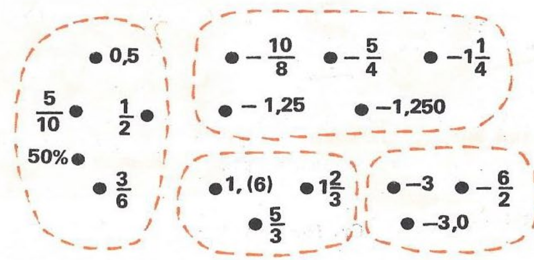
Част от 1; 2

2. Равенство на рационални числа

Да изразим с проценти дробта 0,5:

$0{,}5=\frac{5}{10}=\frac{50}{100}=50\%$.

Фиг. 1



Quiz:

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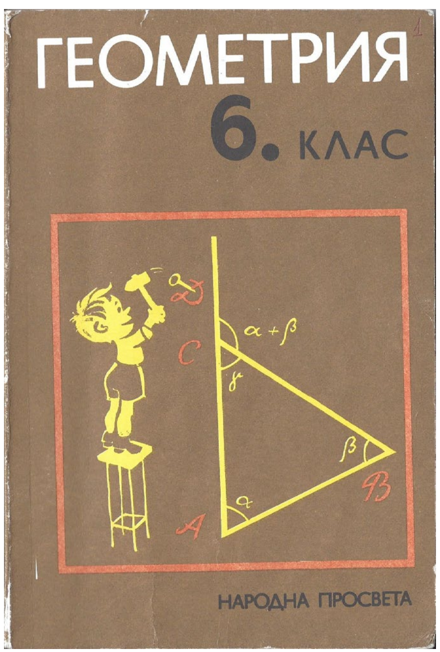
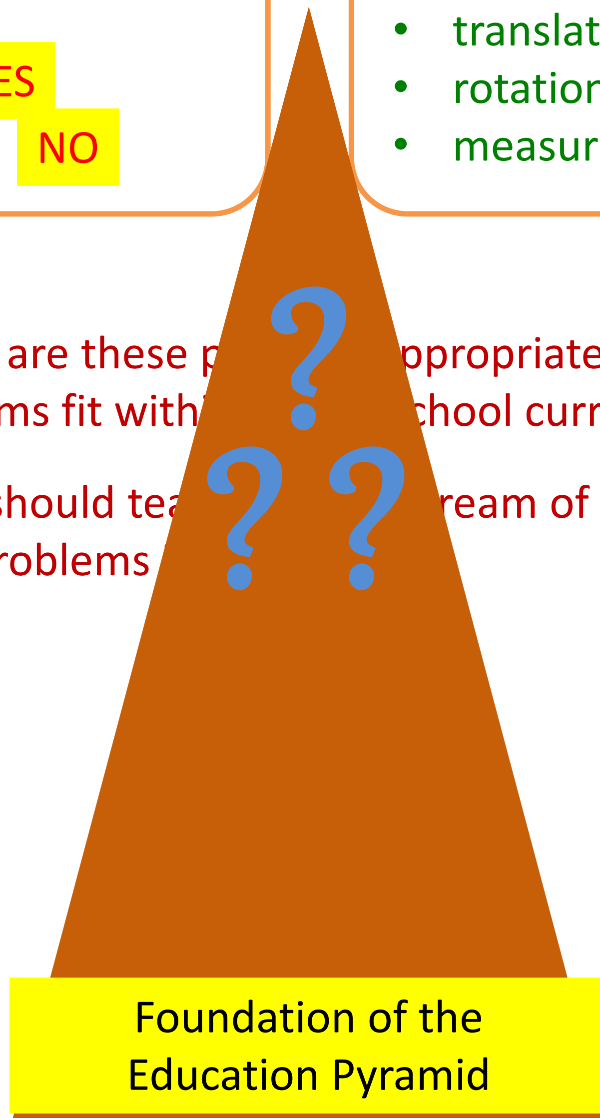
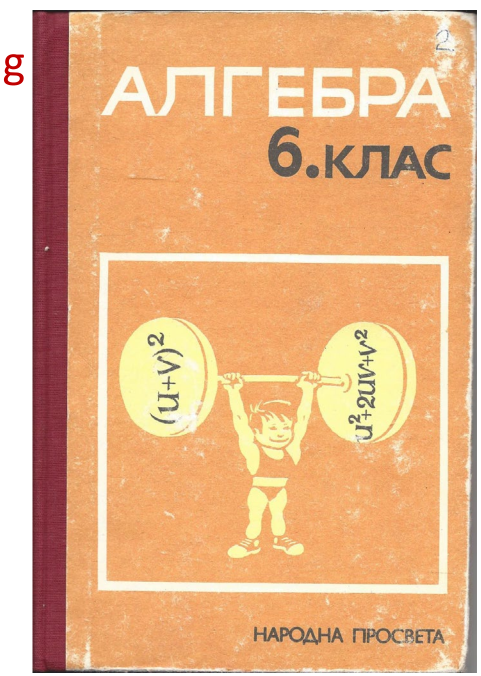
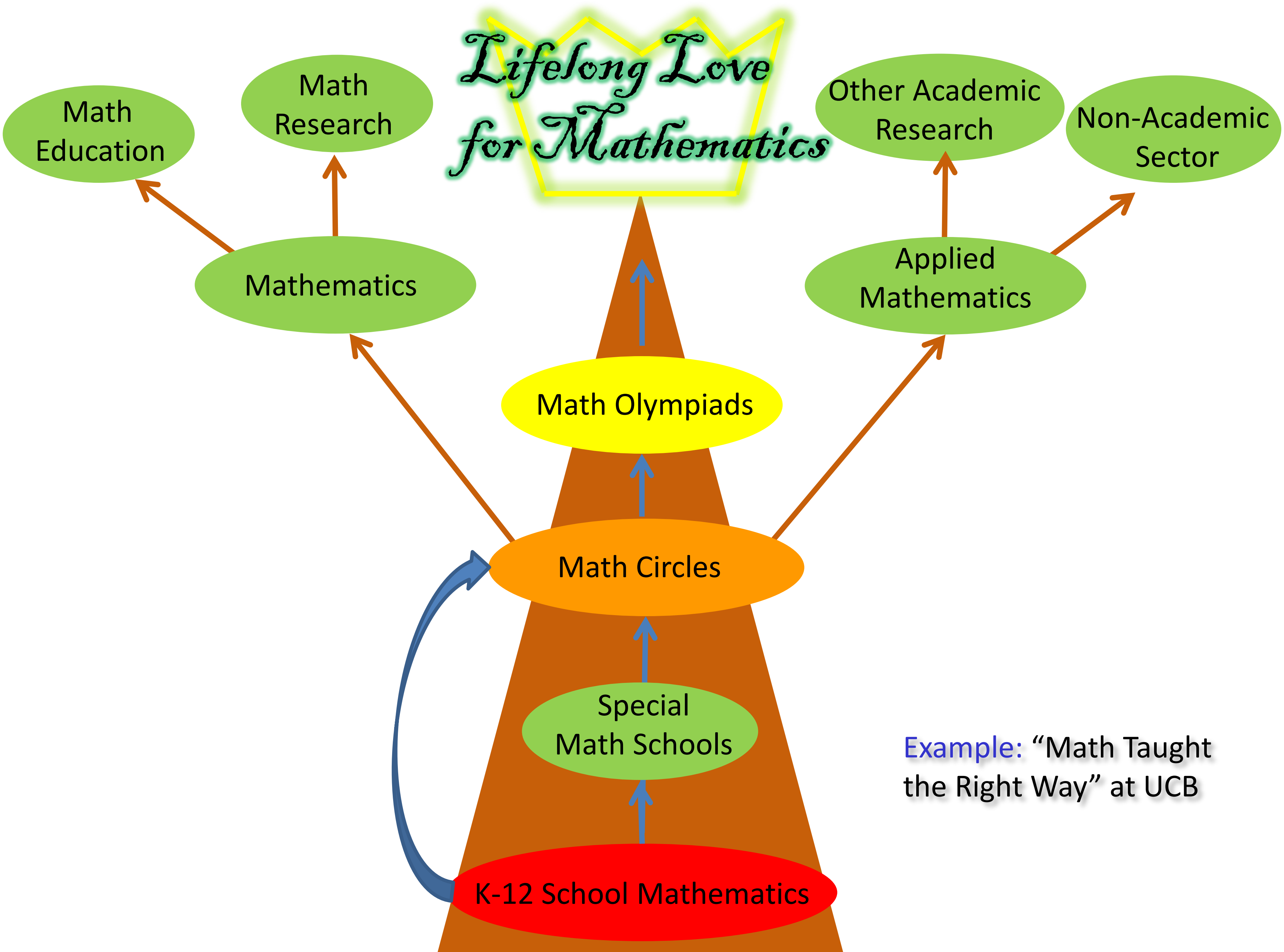


- Can/should tea ream of teaching such problems



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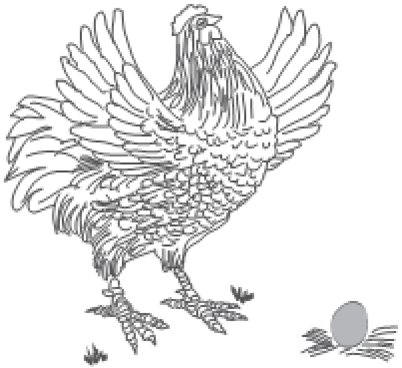
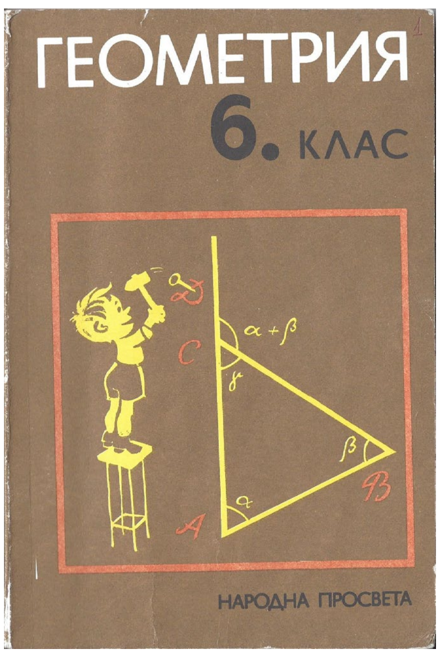
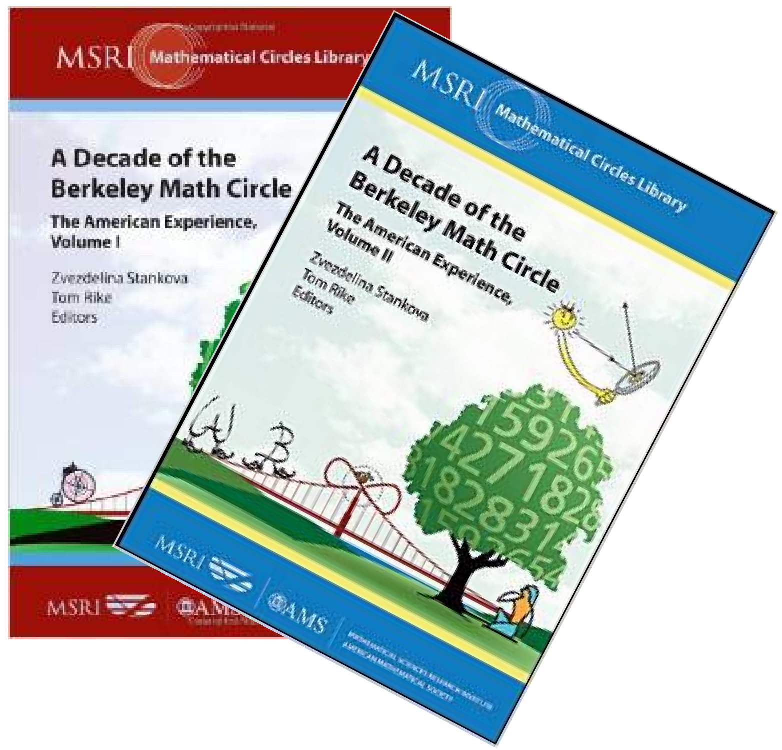
Example: “Math Taught the Right Way” at UCB

School Math Program







U.S. Department of Education National Blue Ribbon Schools 2012

Tehiyah Day School
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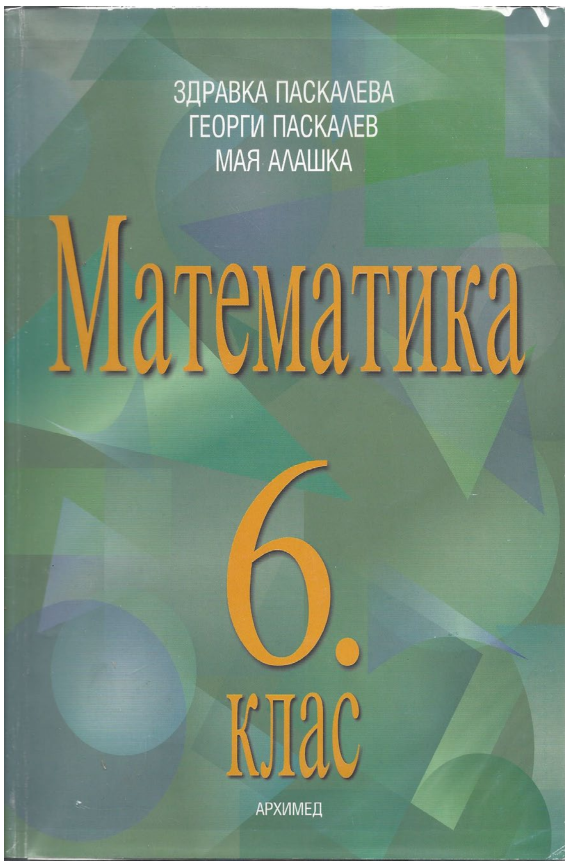


"Math Taught the Right Way" at UCB:
Mondays, 5-8pm
UC Berkeley

Bentley High School
Lafayette, CA

Zdravka Paskaleva, Georgi Paskalev, Maya Alashka

Mathematics

Recommended by

BERKELEY MATH CIRCLE

5 GRADE

Zvezdelina Stankova

Translator and Adapter

Founder and Director

Berkeley Math Circle

Entrance Level

Lessons 1 - 3

3. Entrance Level. Sample Tests and Exams

Test 1

1. The missing digit in $5364 > 5\square95$ is:
   a) 4; b) 8; c) 2; d) 7.
2. The sum $31{,}418 + 9{,}097$ is:
   a) 40,505; c) 40,415;
   b) 40,515; d) 40,405.
3. There are $200,000 in the account of a company. They paid salaries of $105,900. In the account there remained:
   a) $94,000; c) $4,100;
   b) $104,100; d) $94,100.
4. The product of the numbers 1,054 and 68 is:
   a) 10,472; c) 71,642;
   b) 71,672; d) 71,472.
5. There are 864 seats on a train. In each car there are 72 seats. The number of the cars on the train is:
   a) 12; b) 11; c) 14; d) 13.
6. A rectangle has a side of length 39 cm and area 1,248 square cm. The perimeter of the rectangle in centimeters is:
   a) 122 cm; b) 71 cm;
   c) 142 cm; d) 132 cm.
7. The value of the expression $(24{,}125 - 725) \div 3 + 22$ is:
   a) 7,822; b) 936; c) 802; d) 702.
8. The unknown number x in $2{,}102 \cdot 3 - x = 5{,}400$ is:
   a) 1,906; b) 11,706;
   c) 1,006; d) 906.
9. The unknown number x in $(3{,}500 - 1{,}205) \div x = 51$ is:
   a) 53; b) 47; c) 45; d) 43.
10. Decrease the quotient of 4,284 and 21 by 16. The resulting number is:
   a) 8; b) 188; c) 198; d) 4,247.

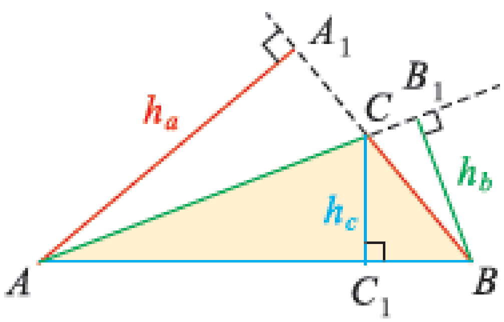
Exam 1

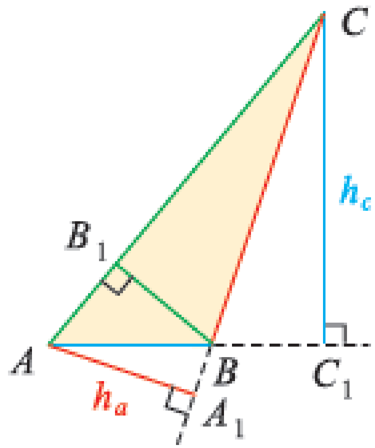
1. Calculate:
   a) $2{,}305{,}162 + 320{,}209$;
   b) $3{,}107{,}105 - 240{,}076$;
   c) $4{,}003{,}103 \cdot 13$;
   d) $32{,}544 \div 16$.
2. Calculate the value of the expressions:
   a) $207 \cdot 5 - 1{,}350 \div 9$;
   b) $38 \cdot 10{,}235 + 62 \cdot 10{,}235$.
3. Find the unknown number x if:
   a) $2{,}503 - x = 45 \cdot 26$;
   b) $37 \cdot x = 11{,}544 \div 26$.
4. There is an amount of $705 for awards. 15 rackets at $23 each and 8 identical balls were purchased. Find the price of one such ball.

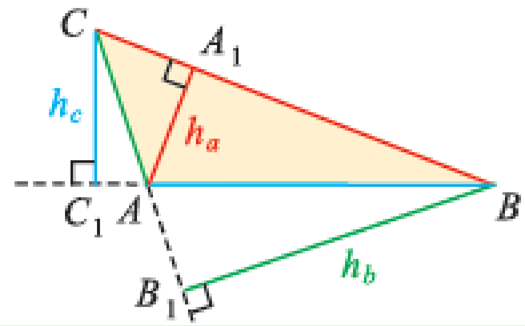
Example 6 Draw altitudes in obtuse ΔABC.

Solution:

I case: $\Delta ACB > 90°$ II case: $\Delta ABC > 90°$ III case: $\Delta BAC > 90°$



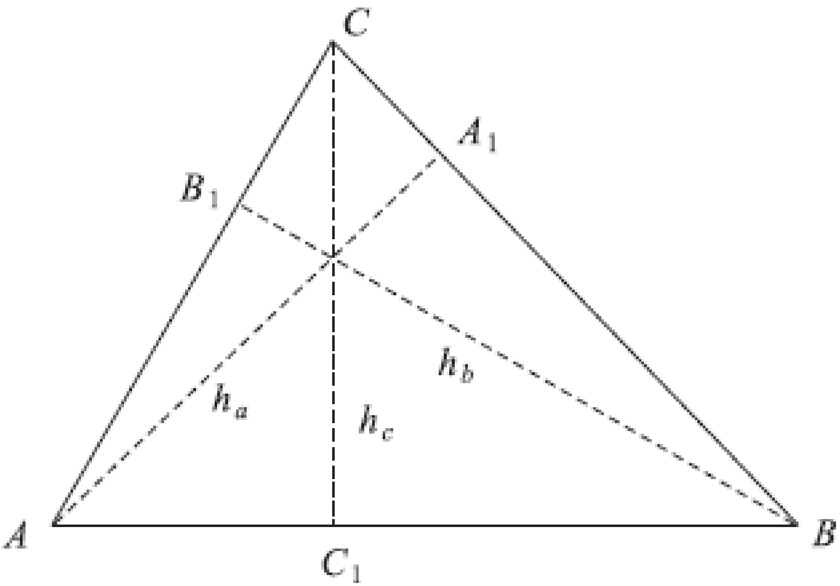




Exercises

1. Draw acute ΔABC and its three altitudes.
2. Draw right ΔABC ($\angle C = 90°$) and its three altitudes.
3. Draw right ΔABC ($\angle A = 90°$) and its three altitudes.
4. Draw obtuse ΔABC ($\angle A > 90°$) and its three altitudes.
5. Draw obtuse ΔABC ($\angle B > 90°$) and its three altitudes.
6. Draw obtuse ΔABC ($\angle C > 90°$) and its three altitudes.

Example 2 (Practical application.) Drawn is ΔABC and its three altitudes. The task is to calculate the area of ΔABC in three different way and to find its average value.



Solve the problem by choosing and measuring the necessary elements of the drawn triangle.

Solution:

We measure with a ruler:

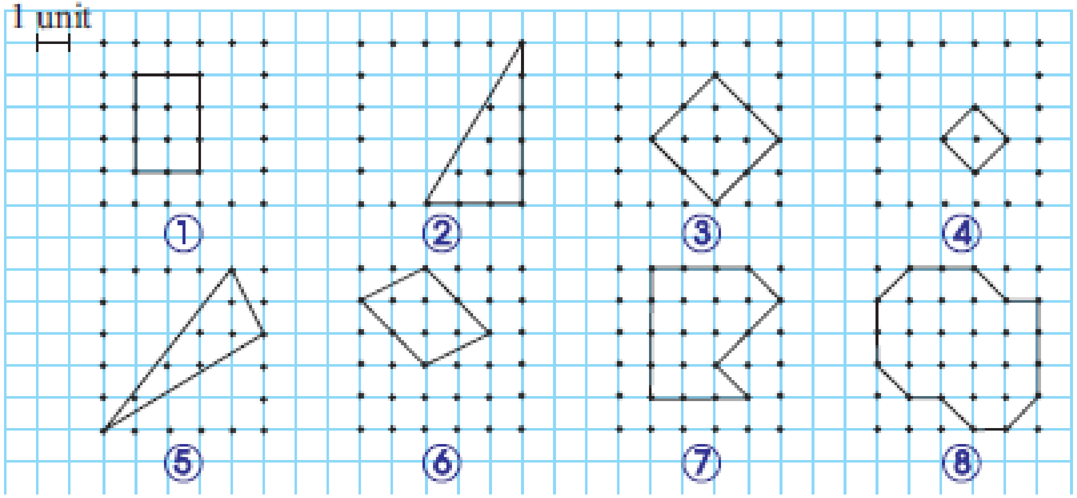
$AB \approx 8$ cm,
$BC \approx 7.2$ cm,
$CA \approx 6$ cm,
and
$AA_1 = h_a \approx 5.8$ cm,
$BB_1 = h_b \approx 6.9$ cm,
$CC_1 = h_c \approx 5.2$ cm.

We calculate the area in three ways:

$a \approx 7.2$ cm, $h_a \approx 5.8$ cm	$b \approx 6$ cm, $h_b \approx 6.9$ cm	$c \approx 8$ cm, $h_c \approx 5.2$ cm
$S = 0.5 \cdot a \cdot h_a$	$S = 0.5 \cdot b \cdot h_b$	$S = 0.5 \cdot c \cdot h_c$
$S \approx 0.5 \cdot 7.2 \cdot 5.8$	$S \approx 0.5 \cdot 6 \cdot 6.9$	$S \approx 0.5 \cdot 8 \cdot 5.2$
$S \approx 20.88$ sq. cm;	$S \approx 20.7$ sq. cm;	$S \approx 20.8$ sq. cm.

The average value is $S \approx (20.88 + 20.7 + 20.8) \div 3 \approx 20.79$, $S \approx 20.79$ sq. cm.

Example 3 Find the area of the figures drawn in the square grid:



The area of the figure in a square grid (*Example 3*) with vertices at the grid points can be found by the following rule (called Pick's Formula):

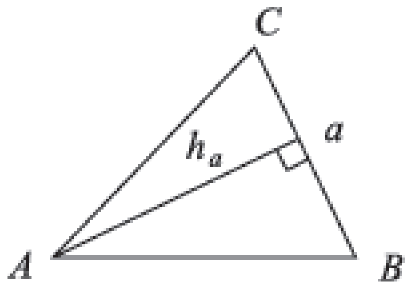
$$S_{\text{figure}} = \boxed{A} \div 2 + \boxed{B} - 1$$

A — number of the points along the perimeter of the figure

B — number of points inside the figure

We will calculate the area of the figures 1 to 8 using this rule:

1. $S = 10 \div 2 + 2 - 1 = 6$ sq. units;
2. $S = 9 \div 2 + 4 - 1 = 7.5$ sq. units;
3. $S = 8 \div 2 + 5 - 1 = 8$ sq. units;
4. $S = 4 \div 2 + 1 - 1 = 2$ sq. units;
5. $S = 3 \div 2 + 6 - 1 = 6.5$ sq. units;
6. $S = 6 \div 2 + 4 - 1 = 6$ sq. units;
7. $S = 14 \div 2 + 6 - 1 = 12$ sq. units;
8. $S = 15 \div 2 + 13 - 1 = 19.5$ sq. units.



ΔABC
$h_a = 58$ mm
$S = 0.435$ sq. dm

$a = ?$

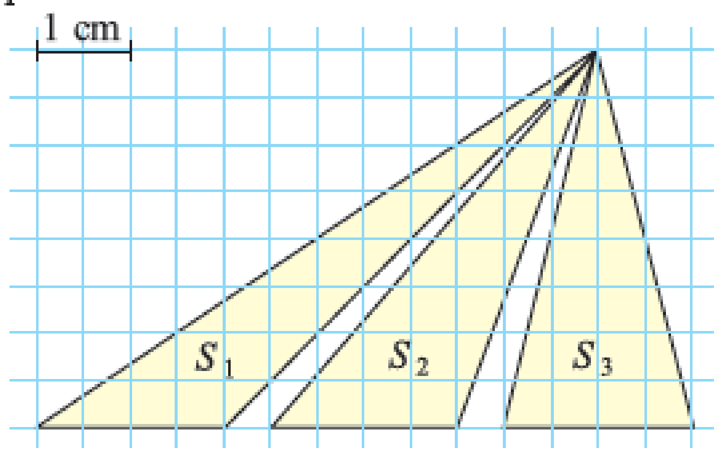
Solution:

$h_a = 58 \text{ mm} = 5.8 \text{ cm}$
$S = 0.435 \text{ sq. dm} = 43.5 \text{ sq. cm}$

$S = 0.5 \cdot a \cdot h_a$
$43.5 = 0.5 \cdot a \cdot 5.8 \rightarrow a = 15$
$a = 15$ cm

Example 5 (Orally)

Find the area of the colored triangles and compare them.

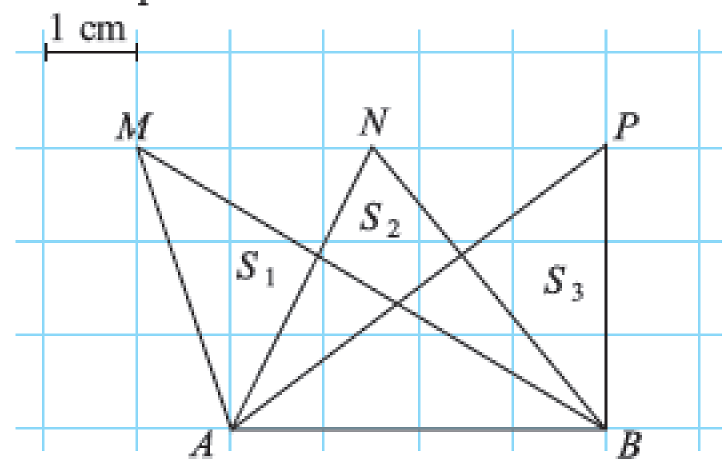


Solution:

$S_1 = S_2 = S_3 = 4$ sq. cm.

Example 6 (Orally)

Find the areas of ΔABM, ΔABN, and ΔABP and compare them.



Solution:

$S_1 = S_2 = S_3 = 6$ sq. cm.

Exercises

1. In ΔABC find "?" in centimeters if given:

a) $S = 12.48$ sq. cm, $c = 78$ mm, $h_c = ?$

b) $S = 21$ sq. cm, $a = 0.7$ dm, $h_a = ?$

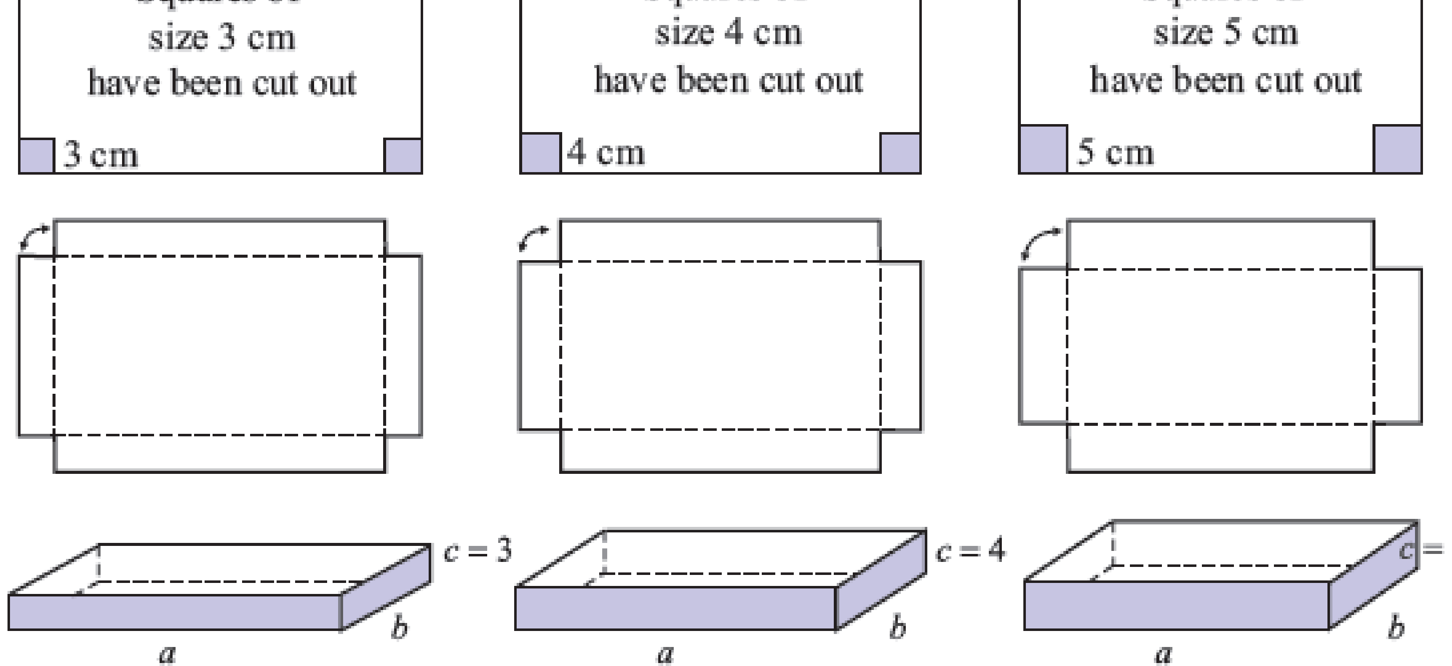
c) $S = 22.8$ sq. cm, $h_b = 0.057$ m, $b = ?$

Find the length of:

a) the other leg;

b) the hypotenuse;

c) the altitude to the hypotenuse.



$c = 3$ cm
$a = 30 - 2 \cdot 3 = 24$ cm
$b = 18 - 2 \cdot 3 = 12$ cm
$V = a \cdot b \cdot c$
$V = 24 \cdot 12 \cdot 3$
$V = 864$ cu. cm

$c = 4$ cm
$a = 30 - 2 \cdot 4 = 22$ cm
$b = 18 - 2 \cdot 4 = 10$ cm
$V = a \cdot b \cdot c$
$V = 22 \cdot 10 \cdot 4$
$V = 880$ cu. cm

$c = 5$ cm
$a = 30 - 2 \cdot 5 = 20$ cm
$b = 18 - 2 \cdot 5 = 8$ cm
$V = a \cdot b \cdot c$
$V = 20 \cdot 8 \cdot 5$
$V = 800$ cu. cm.

We should choose the box made in Project II.

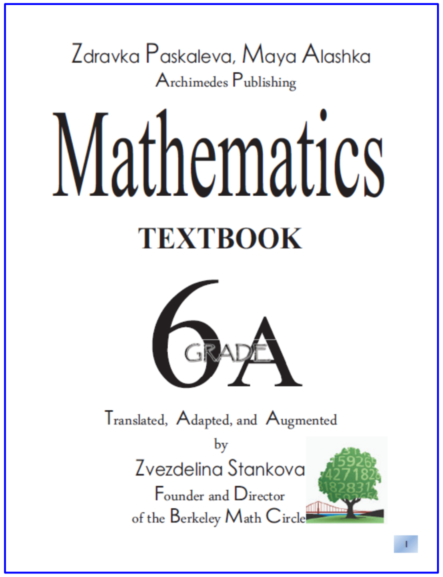
Exercises

1. To repair a volleyball court in the shape of a rectangle with dimensions 18 m and 9 m an extra layer of average width 4 cm was installed on the court. Find how many cubic meters of material were used for this

3. 20 pine boards are in the shape of a rectangular parallelepiped and have dimensions 20 cm, 3 cm, and 4 m.

a) Find out how much the boards weigh if 1 cu. m pine wood weighs 600 kg.

b) If the price of pine wood is $250 per cubic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CONTENTS

5. How Mathematicians Write Solutions

Example 1 Calculate the numerical value of the expressions. Write your solutions in a correct, complete, and clear way. How many points out of 10 do you think you will get?

a) $A = (108 + 37) \div 5 + 31$; b) $B = 2 \cdot a + b \div 3$ if $a = 31$ and $b = 27$.

Correct Solutions:

a) 60;

b) 71.

These are just the *answers*! They are *correct*, but each will receive only 2 out of 10 points.

Warning: Just giving the *answer*, even if correct, will bring very few (if any) points from now on!

Complete Solutions:

a) $108 + 37 = 135$;
$135 \div 5 = 27$;
$27 + 31 = \boxed{58}$.

b) $2 \cdot 31 = 62$;
$62 + 9 = \boxed{71}$.

a) is *partially correct*. There is a carry-over error in the first addition, but the rest is correct (accepting the error). So, 1 out of 2 points for correctness.

The solution for:

a) is *complete*. It shows all steps and calculations. So, 5 out of 5 points for completeness.

b) is *incomplete*. Where did 9 come from? Missing step! So, 3 out of 5 points for completeness.

Total:

a) $5 + 1 = 6$ points. b) $3 + 2 = 5$ points.

b) is *correct*. The calculations (assuming the missing step) and the final answer are correct. So, 2 out of 2 points for correctness.

Words of wisdom: A *complete* solution, even with a calculation error, may bring more points than an incomplete solution with a correct final answer!

Clear Solutions:

a) $A = (108 + 37) \div 5 + 31 = 145 \div 5 + 31 = 29 + 31 = 60$.

b) $B = 2 \cdot a + b \div 3 = 2 \cdot 31 + 27 \div 3 = 62 + 9 = 71$.

These are *perfect solutions*, earning 10 points each! They are **correct** (2 pts), **complete** (5 pts), and **clearly written** (3 pts). Each starts by re-writing the original problem, and then shows all calculations in a logical sequence, using four "=" to connect the different calculations.

A **clearly written** solution is like a *puzzle put together* – one can see each piece in its proper place and can easily observe the connections between the various pieces without having to look for them all over the "table". The **art of writing clear solutions** is a skill developed over years.

Correct – Complete – Clear Rubric

2 points: correct answer

5 points: all steps and calculations

3 points: written in a clear, organized way

CoCoClear: $2 + 5 + 3 = 10$ max

Example 2 Is the solution below to Example 1a) correct, complete, and clear? What do you think: how many points will it receive?

Solution: $A = 108 + 37 = 145 \div 5 = 29 + 31 = 60$.

- The answer is correct. So, 2 points for correctness.
- All steps and calculations are shown, so 5 points for completeness.
- Everything is clearly written. We can follow each step in a logical sequence. So, 3 for clarity.
- It seems this is a perfect solution, deserving 10 out of 10 points! ... **Or is it?**

Alas, three of the equalities are *incorrect*:
- $A \neq 108 + 37$ ($A = 60$);
- $108 + 37 \neq 145 \div 5$ ($108 + 37 = 145$);
- $145 \div 5 \neq 29 + 31$ ($145 \div 5 = 29$).

The actual total score will be 5 points at best:
- *correct*: 2 points (if grader is lenient);
- *complete*: 2 points (this is tough to judge);
- *clear*: 1 point (for writing all on one line).

Yes, we may understand what the writer meant in the solution in Example 2, but this does not make it right! Imagine that you submit an essay in your Language Arts class with the title:

"The siMilleritys an Diferrenses Betwin Matematicle! and Landjaige: Artz Righting"....

Yes, your teacher will likely understand what you meant in this title; but unless you wrote a parody of poor writing, making fun of bad spelling and punctuation errors, your essay will not earn much praise. The same applies to mathematical solutions: the math symbols must be used correctly and different steps must be put together in a logical (not random or most convenient) way.

Example 3 Using the CoCoClear Rubric, grade the solutions below of the equation $16 + x = 35$.

Solution 1:
$16 + x = 35$
$x = 35 - 16$
$\boxed{x = 19}$

Perfect: 10 pts. All "=" are *aligned vertically*; x is in every step.

Solution 2:
$16 + x = 35$
$35 - 16 = \boxed{19}$

Good: $2 + 4 + 1 = 7$ pts. "=" are not *aligned vertically*; x got lost in the second line.

Solution 3:
$16 + x = 35 - 16$
$\boxed{x = 19}$

Mediocre: $2 + 2 + 1 = 5$ pts. Incorrect first line: 16 must not be subtracted like this! Missing step: $x = 35 - 16$.

Solution 4:
$16 + x = 35$
$x = 35 - 16 = \boxed{19}$

Very good: $2 + 4 + 2 = 8$ pts. $35 - 16$ should not be calculated like that on the same line. Missing step: $x = 19$.

Calculating *expressions* is usually written **horizontally** (in one line). Solving *equations* is usually written **vertically** (aligning all "=").

Exercises

Following the CoCoClear Rubric,

1. Solve the following equations:

a) $x \div 18 = 6$; b) $33 - x = 29$;
c) $(24 - 10) \cdot x = 70$; d) $11 \cdot (x + 3) = 121$.

2. Calculate the following expressions:

a) $18 \cdot 3 + 107$;
b) $236 - (13 \cdot 7 + 225 \div 5)$;
c) $(19 - 7 \cdot 2) \cdot (23 - 39 \div 3)$;
d) $17 \cdot a + b \cdot 655$ for $a = 1{,}345$ and $b = 17$.

5. The Operation of Exponentiation with Natural Exponents

The sum of equal summands is written for short as a product of the given summand and the number that shows their number:

$$\underbrace{2+2+2+2+2}_{5 \text{ summands}} = 2 \cdot 5.$$

The product of equal factors can also be written in a short way. We write is as

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{5 \text{ factors}} = 2^5,$$

where | 2 is the number that we multiply, and
| 5 is the number that shows how many factors have been multiplies; it is a natural number.

2^5 We read: "two to the power of 5" or
"two to the fifth power" (or even shorter as "two to the fifth").

The product of n equal factors a, where n is a natural number, is written as a^n and is called **the power with base a and natural exponent n**.

We write $\underbrace{a \cdot a \cdots a}_{n \text{ factors}} = a^n$.

- **The number a is called the base of the power.**
  The base a can be a natural number, zero, or a fraction.
- **The number n is called the exponent.**
  The concept "power" is introduced at first using an exponent that is a natural number.

power $\rightarrow a^n$ $\leftarrow$ exponent of the power (exponent)
$\leftarrow$ base of the power (base)

Examples: $10 \cdot 10 \cdot 10 \cdot 10 = 10^4$; $2.7 \cdot 2.7 \cdot 2.7 = 2.7^3$; $\frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} = \left(\frac{2}{5}\right)^6$; ...

- The exponent is written in a smaller font and is placed to the right and above the base. (The exponent is written as a superscript.)
- If the base of the power is a common fraction or an expression, then it is placed in parentheses.

$a^1 = a$ When there is only one factor a it is customary to write a^1.
Examples: $5 = 5^1$; $10 = 10^1$; $0,2 = 0,2^1$; ...

Example 1 Write as a power the product:
a) $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$; b) $3.5 \cdot 3.5 \cdot 3.5$.

Solution:
a) $\underbrace{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}_{7 \text{ factors}} = 5^7$ b) $\underbrace{3.5 \cdot 3.5 \cdot 3.5}_{3 \text{ factors}} = 3.5^3$

Example 2 Write the powers as products: a) 10^5; b) 0.2^3; c)* $\left(\frac{2}{7}\right)^6$.

Solution:
a) $10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ b) $0.2^3 = 0.2 \cdot 0.2 \cdot 0.2$ c) $\left(\frac{2}{7}\right)^6 = \frac{2}{7} \cdot \frac{2}{7} \cdot \frac{2}{7} \cdot \frac{2}{7} \cdot \frac{2}{7} \cdot \frac{2}{7}$

* We read "two-sevenths to the sixth power".

Example 3 Write, read, and calculate the power:
a) with base $\frac{1}{2}$ and exponent 4; b) with base 1.2 and exponent 2.

Solution:
a) $\left(\frac{1}{2}\right)^4$, one half to the fourth power, $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$;
b) 1.2^2, one point two to the second power, $1.2 \cdot 1.2 = 1.44$.

The operation of exponentiation with a natural exponent
The operation through which we calculate the value of a given power is called **exponentiation**.

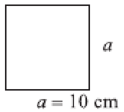
To perform exponentiation means to find the product of n factors, all equal to a.
When $n = 3$ and $a = 2$, we have $2^3 = 2 \cdot 2 \cdot 2 = 8$.
If the base of the power is 0, then the power 0^n is equal to 0.

$0^n = 0$
$1^n = 1$ If the base of the power is 1, the power 1^n is equal to 1.

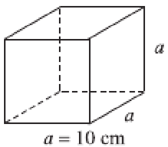
The power | a^2 (a in the second) is also read as "**a squared**";
| a^3 (a in the third) is also read as "**a cubed**".

The units for area and volume are also read as:
square centimeter (sq. cm)*, written also as **cm²**.
cubic centimeter (cu. cm), written also as **cm³**.

Example 4 Calculate: a) the area of a square with side $a = 10$ cm;
b) the volume of a cube with edge $a = 10$ cm.

Solution:
a)


$S = a \cdot a$
$S = a^2$
$S = 10^2 = 100$
$S = 100 \text{ cm}^2$

b)


$V = a \cdot a \cdot a$
$V = a^3$
$V = 10^3 = 1000$
$V = 1000 \text{ cm}^3$

Exercises

1 Write the products as powers:
a) $7 \cdot 7 \cdot 7 \cdot 7$; c) $\underbrace{5 \cdot 3 \cdot 5 \cdot 3 \cdots 5 \cdot 3}_{12 \text{ factors}}$;
b) $\underbrace{3 \cdot 3 \cdots 3}_{20 \text{ factors}}$; d) $\frac{5}{7} \cdot \frac{5}{7}$.

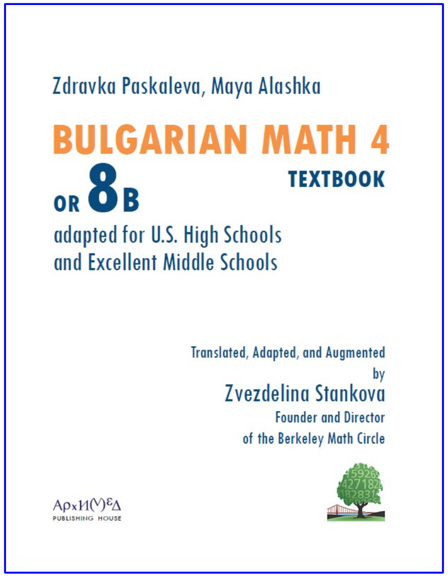
2 Write the powers as products of equal factors and calculate their values:
a) 5^5; c) 2^9; e) $\left(\frac{1}{5}\right)^5$;
b) 6^4; d) 3.2^3; f) $\left(\frac{2}{3}\right)^6$.

3 Perform the exponentiation:
a) 8^3; c) 0.2^3; e) $\left(\frac{2}{3}\right)^4$;
b) 4^4; d) 0.8^2; f) $\left(\frac{2}{7}\right)^3$.

4 Write the next number x if:
a) $1, 4, 9, 16, x$; b) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, x$.

5 We are given the numbers:
a) 48; b) 64; c) 252; d) 360; e) 4,410.
Represent each of these numbers as a product of powers with bases equal to the prime factors of the number.

* In this textbook the units are written according to the International System of Units, the modern form of the metric system.



Let the Math Continue!

146. Circumscribed Quadrilateral. Exercises

Example 1 **Basic Example** We have inscribed a circle in a trapezoid. Prove that the altitude of the trapezoid is equal to the diameter of the circle.

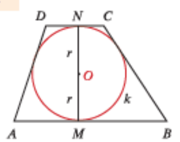


Fig. 108

Proof: (Fig. 108)

We denote the tangent points of k with AB and CD correspondingly by M and N.

From the property of the tangent line:

$OM \perp AB$, $ON \perp CD$. But $CD \parallel AB$ so $ON \perp AB$.

Then the diameter $MN \perp AB$ (CD) and hence

MN is an altitude of the trapezoid ($MN = h$).

We obtain $h = 2r$.

Example 2 Prove that the area of quadrilateral circumscribed about a circle is equal to the product of the semi-perimeter of the quadrilateral and the radius of the circle.

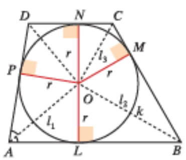


Fig. 109

Proof: (Fig. 109)

$$S_{ABCD} = S_{\Delta ABO} + S_{\Delta BCO} + S_{\Delta CDO} + S_{\Delta DAO}$$

$$= \frac{AB \cdot r}{2} + \frac{BC \cdot r}{2} + \frac{CD \cdot r}{2} + \frac{DA \cdot r}{2}.$$

We used that $OL = OM = ON = OP = r$.

$$S_{ABCD} = \frac{r}{2}(AB + BC + CD + DA) = \frac{r}{2}P = pr$$

$S_{ABCD} = pr$, where $p = \frac{1}{2}P$.

Example 3 **Basic Example** Prove that a necessary and sufficient condition for an isosceles trapezoid to be circumscribed about a circle is its midsegment equals its leg.

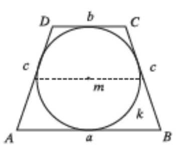


Fig. 110

Proof: (Fig. 110)

We denote the legs by c, the bases by a and b, and the midsegment by m.

1 $ABCD$ circumscribed about k

$\Rightarrow a + b = c + c$

$\Rightarrow a + b = 2c$

$\Rightarrow c = \frac{a+b}{2}$, $c = m$.

2 Assuming $m = c \Rightarrow \frac{a+b}{2} = c$

$\Rightarrow a + b = 2c$

$\Rightarrow a + b = c + c.$ (1)

(1) is a sufficient condition for trapezoid $ABCD$ to be circumscribed about a circle.

Example 4 A trapezoid is circumscribed about a circle. Prove that the circles whose diameters are the legs of the trapezoid are mutually tangent.

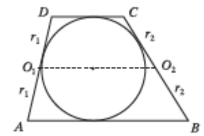


Fig. 111

Proof: (Fig. 111)

The midpoints of the legs O_1 and O_2 are centers of the two circles $k_1(O_1)$ and $k_2(O_2)$. Then O_1O_2 is the midsegment in the trapezoid $ABCD$ and $2O_1O_2 = AB + CD$.

The property of circumscribed quadrilaterals, $AB + CD = AD + BC$

$\Rightarrow 2O_1O_2 = AD + BC.$

$O_1O_2 = \frac{1}{2}AD + \frac{1}{2}BC = r_1 + r_2$

Finally, $O_1O_2 = r_1 + r_2 \Rightarrow$ circles k_1 and k_2 are tangent.

Example 5 Diagonal AC of quadrilateral $ABCD$ divides it into two triangles. The incircles in these two triangles are tangent to AC in points M and N. Express MN by the sides of the quadrilateral.

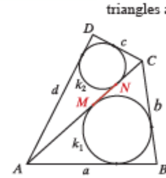


Fig. 112

Solution: (Fig. 112)

We denote the sides of the quadrilateral by $AB = a$, $BC = b$, $CD = c$, and $DA = d$, and the semi-perimeters of ΔABC and ΔACD by p_1 and p_2.

AN tangent to $k_2 \Rightarrow AN = p_2 - c$; AM tangent to $k_1 \Rightarrow AM = p_1 - b$.

Then $MN = AN - AM = p_2 - p_1 + b - c$

$$MN = \frac{d+c+AC}{2} - \frac{a+b+AC}{2} + b - c$$

$$= \frac{b+d}{2} - \frac{a+c}{2} = \frac{(b+d)-(a+c)}{2}.$$

Example 6 A necessary and sufficient condition for a quadrilateral to be circumscribed about a circle is that the inscribed circles for the two triangles into which the quadrilateral is split by any of its diagonals are tangent.

If N is between A and M, we obtain $MN = \frac{(a+c)-(b+d)}{2}$.

Proof: (Fig. 112-113) Recall the notation in *Example 5*.

- If k_1 and k_2 are tangent, then in *Example 5* we have $MN = 0$; i.e., $a + c = b + d$. By T22, $ABCD$ is circumscribed about a circle.
- Conversely, if $ABCD$ is circumscribed about a circle, then by T22 again, $a + c = b + d$. Thus, $MN = 0$ and k_1 and k_2 are tangent.

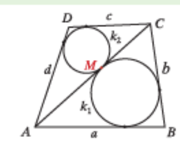


Fig. 113

Exercises

1 Find the diameter of a circle inscribed in an isosceles trapezoid if the midsegment of the trapezoid is 10 cm and one of its angles is 150°.

2 An isosceles trapezoid with acute angle 30° is circumscribed about a circle. The midsegment of the trapezoid is 80 cm. Find the radius of the circle.

3 Consider a quadrilateral circumscribed about a circle with radius 3 cm. The sum of two opposite sides of the quadrilateral is 20 cm. Find the perimeter and the area of the quadrilateral.

4 We have inscribed a circle with radius 2 cm in a trapezoid. Find the area of the trapezoid if the sum of its legs is 10 cm.

5 A right trapezoid $ABCD$ ($AD \perp AB$, $AB \parallel DC$) has $\angle ABC = 30°$ and is circumscribed about a circle with radius 1 cm. Find the perimeter and the area of the trapezoid.

6 Quadrilateral $ABCD$ is circumscribed about a circle $k(O)$. Prove that

$$\angle AOB + \angle COD = 180°.$$

13. Math 4 (8B) Program by Topics and Lessons

Algebra

Functions

- Coordinate systems (Review) L83
- Definition. Attributes L84
- Graph of a Function L85
- Ways to Define a Function L86
- Types of Functions:
  - Direct Proportionality L87-88*
  - Linear F'ns and Inequalities L89-90,94-95
  - Modular Functions L91
  - Inverse Variation L92-93
  - The function $y = ax^2$, $a \neq 0$ L96-97

Systems of Equations

- Linear Eq'ns w/ Two Unknowns L102-103
- Systems of Linear Equations L104
- Solving Systems by:
  - Substitution L105
  - Addition L106
  - Other Op's L107

Systems of Inequalities

- Set-up, Define, and Solve L112-114
- $f(x) \cdot g(x) \gtrless 0$ L115
- $|ax + b| \lessgtr c$ L116-117

Equations and Roots

- Factoring Quadratics L151
- Biquadratic Equations L152
- Eq'ns Reducing to Quadratics L153-154
- Vieta's Formulas. Applications L155-156

Rational Expressions

- Rational Expressions. Types L159
- Basic Property L160
- Putting over a CD L161
- Operations L162-164
- Rational Equations L165-166
- Modeling with Rational Equations L167

Combinatorics

- Sets. Compounds. Types. Rules L171
- Multiplying/Adding Possibilities L172
- Permutations L173
- Variations L174
- Combinations L175

Geometry

Circles and Angles

- Circles, Points, and Lines. Definitions L121
- Tangents. Two Circles L122-123
- Angles in Circle:
  - Central Angle, Arc, Chord L124-125
  - Inscribed Angle L126
  - Peripheral Angle L127
  - Applications of Angles L128-129

Circles and Polygons

Construction Problems:

- Elementary and Basic Constructions L132
- Ingredients of Construction Solutions L133
  - Geometric Locus of Points (GPL) L134
  - Overseeing a Segment at an Angle α L135
  - Constructing a Triangle L142

Special Circles and Points for Triangles:

- Circumcircle L136-137
- Incircle L138-139
- Famous Points L140-141

Quadrilaterals and Circles:

- Circumcircle/Cyclic Quadrilateral L143-144
- Incircle/Circumscribed Quadrilateral L145-146
- Applications of Special Circles to Geometry Problems L148

Geo-Algebra

Functions, Equations:

- Parallel Lines L89,102-104
- Graphing and Drawing L88-91
- Vertical Lines and Line Test L85,89
- Areas, Graphs, and Functions L92,100-101

Circle and Vector Geometry:

- Areas, Circles, and Algebra L138-139,145-146
- Vectors and Algebra L185-186

Geometry in Algebra/Number Theory:

- Areas and Roots L157
- Alg. and Geom. Inequalities L178

Applications

Functions, Systems, Rational Expressions:

- Wood/Tank/Water Problems L87,91,99
- Body Mass Index, Heating Elements L87,90
- Problems on Alloys, Work, Atmospheric Pressure, Motion L92,108,110,111,167-169

Combinatorics:

- Coding L173,177
- Number of Games, Lines, Diagonals L174-177
- Number of Divisors L172,177

Algebra:

- Applications of Vieta's Formulas L156-158

Optimization:

- Extreme Values of Functions L97,99

Legend: Algebra, Geometry, Applications, Geo-Algebra, Logic and Proofs

* L87-88 – Lessons 87 through 88

D A T E B

Concepts and Logic

Basic a Logic Structure of Mathematics:

- Points vs. Pairs of Coordinates in the Plane L83
- Conjecture vs. Mathematical Proof L88
- Impossibilities: Vertical Line Test L85,89
- Domains: Implications for Functions L86-87,92
- Change in Definitions and Consequences: L184
- To Solve an Equation or Inequality: to Find the Solutions or Determine There are None L152
- Alternative Proofs and Converse Theorems L155
- Unique, Exactly One, No Object L85,104,136

Build a Logic Network in Circle Geometry:

- Circles, Secants, and Tangents L121-123
- Circles, Chords, and Arcs L124-125
- Central, Inscribed, Peripheral Angles L124-129
- Circumcircles and Incircles of Triangles and Quadrilaterals L136-139,143-146
- Famous Points in Triangles L140-141

Build a Logic Network in Geometric Constructions:

- Elementary and Basic Constructions L132
- Solving Construction Problems L132-133
  - Analysis, Algorithm, Proof, Discussion
  - Position-Free vs. Positional Problem
  - Same solution "Up To Congruence"
- Geometric Locus of Points L134-135,142
  - GLP that Oversee a Segment at an Angle
  - Constructing a Triangle

Build a Logic Network in Functions:

- Inputs, Values, Domains, Graphs L84-85
- Types/Graphs: Direct/Inverse Variations, Linear, Parabolas, Hyperbolas, Modular L84-97
- Connect Functions, Equations, and Inequalities through the Graphing and Geometry L94-95

Build a Logic Network in Systems:

- Systems of Two Equations w/ Two Unknowns:
  - Linear Equations and Their Graphs L102
  - Equivalent Systems L102
- Systems of Two Inequalities w/ One Unknown:
  - Unions/Intersections of Intervals L113-117
  - Product and Modular Inequalities L115-116

Build a Logic Network in Equations and Roots:

- Quadratic Formula and a Shortcut L151
- Factoring Trinomials L151
- Vieta's Formula and a Shortcut L155-156
- Equation Reducing to Quadratics L152-154

Build a Logic Network in Combinatorics:

- Operations with Possibilities L171-172
- Permutations L173
- Variations L174
- Combinations L175

Problem Solving Techniques

Use Finitely Many Points to Draw Conclusions:

- In Tables for Functions L84
- In Making Predictions about Next Points L85

Multiple Solutions in:

- Construction Problems L133,135
- Geometric Proofs L143
- Problems with Equations/Functions L105,156

Reduce to Previous Problems/Situations:

- Repeated Expressions L152-155
- in Circle Geometry L123,139-141,143,146-147
- in Algebra L151,161-162,166

Split into Cases when Solving and/or Proving:

- Signs of Roots of Equations L156
- Circle Geometry L122,126,128,134-136,143
- Rational Expressions L162,166-167
- Combinatorics and Number Theory L172

Organize Data/Knowledge by Making:

- a Table L84-99,102-104,107-108,139,156,168, L178,180,181,184,187
- a Graph L84-99,102-104,107,178,182-184
- a List L132,140,157,159,179,187
- a Diagram L84,98-99,106,108,112-118, L171-172,176,179,181-185,187,190
- a Collection of Pictures L98-99,109,112,118, L121-148,178,182-183,185-189

Create and Write Proofs

- "If and Only If" L121,125,144-146
- By Contradiction L85,126
- Extra Construction L129,141,148
- Identifying Combinatorial Type L176
- Algorithms L105-106,129,132-135,142,161

Study and Review

Review: L83,132,178-190
Summarize: L98-99,109,130,147,157,168
Chapter Tests: L100,110,131,150,158,169,177
Problems: L101,111,120,149,170
Exit Tests: L191

8.5. A Reality Check: Geo-Algebra in the U.S.?!

It is unrealistic (although not entirely impossible) to imagine that the pre-college curriculum in the U.S. can quickly change so much that it will embrace Geometry as an equal counterpart of Algebra, will split textbooks and classes into Algebra and Geometry ones, and will employ *the Geo-Algebra symbiotic model*.

The present textbook series capitalizes on a compromise between Eastern European and U.S. math educational systems and realities.

On the other hand, times have changed. *The Bulgarian curriculum has moved part-way towards the U.S. one*; e.g., newer Bulgarian textbooks feature both Algebra and Geometry, while preserving the mathematical integrity and pedagogical insights of a tremendously successful curriculum from the near past.

At a Glance:

The New Middle School Curriculum Adapted to the U.S. Reality by Topic and Semester

6A Textbook
- *Decimals* operations, algebra w/ numbers; efficient calculations; motion & word problems
- *Divisibility* criteria for divisibility; *GCD* and *LCM*
- *Plane Geometry* altitudes; parallel lines; quadrilateral: perimeter, area; apply algebra
- *Solid Geometry* rectangular parallelepiped: surface area, volume; applications
- *Write:* correct, complete, and clear solutions.

6B Textbook
- *Fractions, %'s* operations; part of a whole; applications
- *Exponents* rules; properties
- *Rational #'s* operations with pos./neg. numbers; absolute value; equations
- *Graphing, Data* coordinates, diagrams/tables, arithmetic mean
- *Plane Geometry* circles, discs, polygons: perimeter, area; regular n-gons; constructions
- *Recognize:* expressions and equations.
- *Split:* solutions into "given," "to find," and calculations.
- *Draw:* pictures and diagrams.

Math 1 — 7A Textbook
- *Proportions* properties, applications
- *Polynomials* variables, numerical values, degrees, operations, algebraic identities; factoring in degree 2
- *Solid Geometry* types of prisms and pyramids; surface area, volume; constructions, applications
- *Include:* relevant formulas with variables; then substitute given numerical data.
- *Apply:* algebra tools to solve geometry problems.

Math 2 — 7B Textbook
- *Equations* linear, factored quadratic, parametric; w/ absolute value; modeling, practical applications
- *Plane Geometry* basic notions: pairs of angles, perpendicular/parallel lines; criteria, axioms, theorems
- *Congruent Δ's* criteria, isosceles Δ, perpendicular and angle bisectors, altitudes, medians
- *Understand:* the structure of mathematics.
- *Start proofs:* using definitions, primary concepts, axioms, and theorems.

Math 3 — 8A Textbook
- *Alg. Inequalities* properties, one variable, linear; inequalities vs. equations
- *Square Roots Quadratic Eq'ns* irrational numbers; algebraic expressions; quadratic f-la
- *Parallelograms Trapezoids* properties, criteria, proofs, constructions, applications
- *Intro to Vectors* operations; applications
- *Isometries:* translations, rotations, central symmetries, reflections
- *Devise:* algorithms for geometric constructions.
- *Link:* facts with given info.

Math 4 — 8B Textbook
- *Functions* graphs; linear, modular, quadratic functions; applications
- *Systems of Eq'ns and Inequalities* linear w/ one or two unknowns, parameter, abs. value, modeling
- *Rational Expressions* operations, equations, applications
- *Combinatorics* permutations, variations, combinations
- *Circles and Angles* tangents & lines; two circles; arcs, chords & angles; applications
- *Determine:* the hypothesis and what to prove.
- *Construct:* a proof as a logical sequence with words and math symbols.

Optional Topics
- 7A: *Algebra of Polynomials* alg. identities; degree 3 factoring
- 8B: *Equations and Roots* Vieta's formulas; Bi-quadratic eqn's
- 8A: *Geometric Inequalities* between sides; Δ Inequality
- 7B: *Euclidean Constructions* ruler & compass
- 8A: *Vectors* midsegments & centroids
- 8B: *Circles and Polygons* geometric loci; circumcircles and incircles; famous pts in Δ
- *Find:* multiple ways to solve.
- *Distinguish:* between faulty and valid reasoning.
- *Learn:* from own experience and others' solutions.

Legend
- *Algebra*
- *Geometry*
- *Number Theory, Combinatorics, Statistics*
- *Logic and Proofs Techniques*

7. A Cultural Shift

Several features of the textbooks, which are typically ignored or de-emphasized in standard U.S. middle school curricula, represent a *cultural shift* in how mathematics is viewed and studied in the U.S., but which are very well-known around the world and have been adopted by many countries for decades, if not centuries.

7.1. Reading Mathematics. Every math problem has "words" – these are concepts expressed in different ways, whether as numbers, standard everyday words, or by visual aids such as diagrams and figures. Emphasizing the different forms of "words" as a regular part of math language and math communication and *learning to read, interpret, and write in "words"* is a major part of the new curriculum. The pilot parent's comments below addresses this and more:

A pilot 6th grade parent shares, 7 months after the new curriculum was launched:

"As a teacher and teacher educator, I am familiar with several math programs that have been used throughout the past twenty years in California. The Bulgarian math program is exceptional in the way it supports students' deep understanding of mathematical concepts, develops students' ability to transfer skills to new and unfamiliar mathematical situations, and creates a mathematical language for students to express themselves mathematically, using numbers, drawings, and narratives.

I can see, and most importantly hear, the way my child interacts with the Bulgarian math program and I am impressed with the strong foundation it is giving him that will surely lead to lifelong competency and the ability to pursue more advanced mathematics.

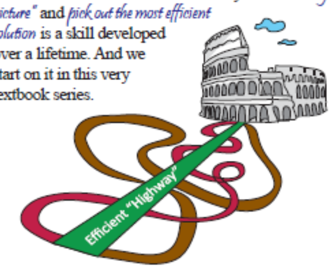
Jacqueline Regev, parent and educator

7.2. Logic and Communication. Learning to know *correct from flawed reasoning* and being *able to explain* one's mathematical ideas smoothly and convincingly to others will represent a gradual move towards *a more mature relationship with mathematics*.

7.3. Writing Mathematics. "Showing your work" is only the beginning to understanding how math really works. All problems have answers in the back of the textbook. Bringing back just an answer on a homework problem from the textbook will be worth no credit. *The explanation that leads to the answer* will be what counts in our study of middle school mathematics. Even harder than learning to correctly interpret problems is learning to consistently *write solutions in a correct, complete, and clear way.* (See Lesson 5 in the 6A textbook.)

7.4. Multiple Solutions. Although there is usually (but not always!) only *one* correct answer to a math problem, the beauty of mathematics is that there may be *different solutions leading to that answer.* These are not "subjective opinions"; rather, they are objective mathematical creations that obey the laws of Logic. Students will learn that each solution (and even each incorrect attempt) has its value and usefulness in the long run and that one needs to *open up to others' ideas and ways of thinking* as a path to enriching oneself, to becoming more proficient and, ultimately, wiser.

7.5. Efficient Solutions. Yet, among "all the roads [that] lead to Rome," there might be a shortest or an easiest to follow. The ability to *see "the big picture"* and *pick out the most efficient solution* is a skill developed over a lifetime. And we start on it in this very textbook series.



6. Relationship with Mathematics

With some notable exceptions, especially in Geometry, the actual math content of these textbooks does not "look" much different from just about any standard U.S. middle school textbook. After all,

A *linear equation* $5x - 7 = 11$ looks just like a linear equation, no matter if it is displayed in a U.S., Uruguayan, or Bulgarian textbook.

The *Pythagorean Theorem* will still say $a^2 + b^2 = c^2$ for the two legs and the hypotenuse of a right triangle.

A shortcut to finding out if a number can be *divided by 3* (without remainder) will still be to check if the sum of its digits is itself divisible by 3.

... because *mathematical truth is universal.*

Beyond the content, two *fundamental questions* that a good teacher must ask about any textbook are:

Question 1: How are the math facts *presented?* In what order and what style? Is there a balance between computational (algorithmic) parts and theoretical (abstract) parts?

Question 2: What *relationship* between the student and the material is developed? The deeper the relationship, the more lasting and more useful the math knowledge.

This textbook series creates the environment for a *strong bond between the student and the material,*

by a *gradual climb,*

- in the *difficulty and breath* of math content;
- in the expectations of *quality, depth, and rigor* of student work;

by building a robust *Temple of Mathematics,*

- always *depending on previous knowledge* and reinforcing it via new applications and without needless repetition;
- always looking ahead and *linking current material to new things* to discover.

At any time throughout the school year the place and importance of the concepts, ideas, and techniques studied are *clear within the "bigger picture" of Mathematics.* Unfortunately, just as described on the previous pages, meddling with textbooks and curricula is a wide-spread phenomenon in the U.S. middle school math education! Here are four sure ways to ruin the structure and purpose of any math textbook, whether good or bad:

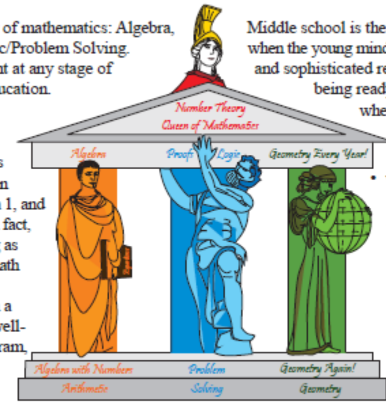
1. Omitting sections.
2. Changing the order of the material in the textbook.
3. Cluttering lessons with handouts from other curricula.
4. Sacrificing depth and teaching to the answers!...

Remember that no one wants to climb up a flimsy structure! As a result, over time the top floors of the Math Temple will become uninhabitable if the actions in the red box run amok in math lessons.

Instead, the deep thought and math wisdom put into creating these textbooks must be respected, or there is no point in starting with this program. Thus, the present textbooks must be *followed just as systematically* as one studies a new language, violin, or chess.

8. The Pillars of Mathematics

There are *three pillars* of mathematics: Algebra, Geometry, and Logic/Problem Solving. They must be present at any stage of pre-college math education.



8.1. Algebra*

There is so much fuss to distinguish between Pre-Algebra, Algebra 1, and Algebra 2... when, in fact, there is no such thing as "Pre-Algebra" as a math subject, nor are there "Algebra 1 or 2"! In a well-designed and well-executed K-12 program,

- *Arithmetic*

  gradually turns into

- *Algebra* $\xrightarrow{\text{naturally}}$ leads to
- *Real Analysis.***

Alas, the typical U.S. student:

- hops along uneven and vaguely defined passages between Pre-Algebra, Algebra 1, Algebra 2, Pre-Calculus, and Calculus;
- experiences a chop-chop approach of poorly matched Algebra curricula from grade to grade;
- is taught to be more concerned about memorizing algorithms*** and passing test benchmarks,

than what is truly important:

- *making connections* and fitting the many pieces together in a giant "jig-saw puzzle" to see the *"big picture";*

and little by little,

- building a *solid Pillar of Algebra* in his/her Temple of Mathematics.

Middle school is the time when math clicks: when the young mind can handle logical leaps and sophisticated reasoning. Yet, instead of being ready for the "big moment," when middle school knocks on the door, teachers and parents alike start:

- worrying about students *"missing algebra skills";*
- pulling the class back to arithmetic operations because all of a sudden *"pre-algebra" is too hard;*
- going in circles;
- losing time, energy, and confidence; and
- ultimately, *weakening the math structure.*

8.2. The Missing Algebraic Link

Why is this happening? The major Algebra mishaps in U.S. middle schools are due to events that have already occurred in elementary school, and are NOT necessarily due to poor arithmetic skills. Expecting that a middle school student will leap from *arithmetic operations with numbers,* e.g.,

$$97 \cdot 2016 = 195{,}552,$$

to *algebraic operations with symbols,* e.g.,

$$(a + b)\,(a - b) = a^2 - b^2,$$

is asking for trouble. There is a whole array of missing intermediate steps, to which we refer as *"Algebra with Numbers"* and which can and should be started very early in the learning process.

How early? On the next page we list some examples from *early grades in Bulgaria.* They also represent well what happens in the algebraic progression in other countries around the world.

* Algebra was present in ancient Greek mathematics mostly as *geometrical* algebra. The word "algebra" is Latin for the Arabic "al-jabr," or "the reunion of broken parts". In our "divine" interpretation of mathematics, the Greek mathematician *Diophantus, the Father of Algebra,* is supporting the math structure on the left and is the only non-deity in the drawing; yet, as a mortal, he had tremendous world influence in both Algebra and Number Theory.

** a.k.a. *Calculus* in first 2 years of U.S. college education.

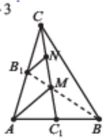
*** Automated "recipes" for some types of problems.

9.1. Examples of Scaffolding

The 8B Textbook features the following exercises from the Chapters on *Algebraic/Geometric Inequalities* and *Parallelograms and Trapezoids* that reach all parts of the difficulty spectrum:

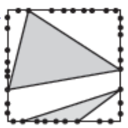
1 (*Basic*). Solve $\frac{x}{x-3}-\frac{54}{x^2-9}=\frac{9}{x+3}$.

2 (*High Proficiency*). Points N and M on median CC_1 of ΔABC are such that $CN=NM=MC_1$. If B_1 is the midpoint of side AC, prove that B, M, and B_1 lie on one line and $AM \parallel B_1N$.



3 (*Hard, but doable with school material*). For which values of the parameter a do the graphs of $f(x)=(a-x)x+(x-2)^2$ and $g(x)=3(x-2)+7$ intersect in a point on the x-axis?

4 (*More advanced material/ideas*). Given a square w/ 9 points on each side, or a rhombus w/ 12 points on each side, how many Δ's are formed by these points?



Occasionally, the textbook uses the eye-icon to signify a fact or technique that is *beyond the textbook material*. For example, an **Alternative Proof** in Lesson 155 challenges the reader:

T2 **Theorem (Vieta):** If a quadratic equation $ax^2+bx+c=0$ has roots x_1 and x_2, then the sum and the product of the roots satisfy:

$$x_1+x_2=-\frac{b}{a}, \quad x_1x_2=\frac{c}{a}$$

Alternative Proof: The trinomial factors as

$$ax^2+bx+c=a(x-x_1)(x-x_2).$$

Expanding and regrouping the RHS, we obtain:

$$ax^2+bx+c=ax^2-a(x_1+x_2)x+a(x_1x_2).$$

The two polynomials on the LHS and RHS are equal for *all* values of x. Hence, their corresponding coefficients must also be equal; i.e.,

$$b=-a(x_1+x_2) \text{ and } c=a(x_1x_2).$$

Solving for the sum and product of the two roots, we obtain $(x_1+x_2)=-\frac{b}{a}$ and $x_1x_2=\frac{c}{a}$.

9.2. What are the *Optional Topics*?

This U.S.-adapted program starts in 6th grade with the 5th grade Bulgarian material; that is,

- In the beginning, it lags a *full year* behind its Bulgarian counterpart.
- By the end of 6th grade, it lags only *half a year*.
- By the end of 8th grade, the difference is cut down to only a *third of a year*.

Almost all of the original 5th-8th grade Algebra material and most of the original Geometry material is covered through the current 6th-8th grade textbooks.

The harder parts of the original 7th-8th grade material on *Degree 3 Polynomials, Vieta's Formulas, Geometric Inequalities, Euclidean Constructions, Midsegments and Centroids, and Circles and Polygons*, are optional and intended *for the more advanced students* as **elective 7th-8th grade topics**. The optional geometry topics are considerably ahead of the standard U.S. high school material and they will provide:

- *a superb preparation* for those who would like *to be challenged and to excel in Geometry.*

9.3. Does the Scaffolding Work?

Does the program really benefit all levels of students? Here is some anecdotal evidence.

Answer 1: A month after the start of the pilot 6th grade program, the teacher received a letter from a parent of a student who was struggling tremendously in the beginning, not participating at all in class, and submitting practically blank tests:

"Thank you so much for this feedback! She is struggling to catch up. I feel like she is learning more math now in one month than she did all last year. Thank you for your help!"

6th grade pilot parent

The student gradually progressed to very good performance both on tests and in class, showing enthusiasm and understanding of material to which she had not even been exposed before but was now fired up to learn.

11. Can a U.S. Middle or High School be Successful with this New Math Curriculum?

What kind of question is this?! Let us analyze the situation as a mathematician would, putting aside our own ambitions and de-politicizing math education.

11.1. The U.S. Gene Pool

Of course, on the average, U.S. students can do just as well as (if not better than) their Bulgarian peers, and, for that matter, any peers around the world. As far as the "gene pool" of the U.S. is concerned, it is probably the best mixture in the world.

The cream of the crop among U.S. students have performed at the top level and have occasionally made the number 1 team in the world (!) at the International Mathematical Olympiads (IMO)*, the preeminent world math competition for pre-college students that tests **not** speed but *depth and originality of mathematical thinking.*

The new math program emulates:

- the IMO spirit of deep and rich math culture
- a mature attitude towards mathematics
- dedication to long-term goals

It is true that very few students will end up being math Olympians, and we will not give false hopes that the current program is training students for the IMOs. However, the expectation is that:

Everyone who is touched by this program will build a solid Temple of Mathematics, where later, if desired, we may hang "fancy chandeliers" and "brilliant art pieces." More importantly, the personal connection with mathematics will remain all through life to be enjoyed and cherished.

11.2. Can Girls Succeed in Math?

Upon coming to the U.S., I was shocked by this question. I was raised in a provincial town in Bulgaria and made it as a high school student to two IMOs, winning silver medals. There was another girl on the Bulgarian team for the 2 years I was there.

It took the U.S. 25 years of participating at the IMOs before the first U.S. girl, Melanie Wood, qualified for the IMOs. I was privileged to train the U.S. national team when Melanie competed at the IMOs, also earning two silver medals. Later, I participated in the training of the other two U.S. girls who went on to win gold medals at the IMOs.

To cut the long story short: anything is possible and *gender does not matter as far as mathematical talent and success are concerned.*

Gender apparently matters in the social and cultural aspects of education, and this severely handicaps many U.S. girls who might otherwise have become excellent mathematicians.

Decades after competing at the IMOs, I learned that Bulgaria is the top country in the world in sending a total of 21 girls so far** to the IMOs. Germany and Russia are next, with 19 and 15 girls, respectively. The USA has only 3 girls so far.

Perhaps *it is time for the U.S. educational system to follow the example of other programs from around the world* that have been hugely successful in raising gen-

* The IMO takes place every summer in a different country. Each country's team has 6 students. As opposed to standardized tests, there are only 6 problems over 2 days, 4.5 hours each day. The solutions are written as rigorous mathematical proofs.

** See *"Cross-Cultural Analysis of Students with Exceptional Talent in Mathematical Problem Solving"* by Titu Andreescu, Joseph A. Gallian, and Jonathan M. Kane, http://www.ams.org/notices/200810/fea-gallian.pdf.

BERKELEY
MATH CIRCLE

REPORT CARD

Student: Ron Weasley

7A group at MTRW Fall 2018

I. HOMEWORK AND EXAM SCORES

I. 7A ALGEBRA Instructor: Elena Blanter	Student % of max score	Class Statistics median	mean	top
1. Overall Homework	85%	68%	57%	97%
2. Pretest Regular Score	98%	77%	71%	100%
3. Pretest Bonus Score	0%	48%	44%	100%
4. Quiz Average Regular Score	53%	48%	45%	86%
5. Final Exam Regular Score	79%	72%	59%	100%
6. Final Exam Bonus Score	50%	0%	16%	100%
7. Overall Bonus	8%	8%	17%	85%
8. Grand Overall (60% #1 + 5% #4 + 30% #5 + 5% #7)	78%	63%	55%	94%

II. 7A GEOMETRY Instructor: Harry Main-Luu		Class Statistics median	mean	top
1. Overall Homework	71%	54%	46%	92%
2. Pretest Regular Score	7%	13%	18%	100%
3. Unit 1 Exam Regular Score	64%	43%	45%	101%
4. Unit 2 Quiz Regular Score	20%	20%	25%	93%
5. Final Exam Regular Score	56%	44%	45%	76%
6. % Improvement (max(#5,#3) – #2)	57%	37%	36%	85%
7. Overall Bonus	5%	5%	7%	40%
8. Grand Overall (30% #1 + 30% (#3 + #5) + 5% #7 + 5% max(#2,#4))	58%	41%	41%	77%

III. 7A Problem Solving Instructor: Elysée Wilson-Egolf		Class Statistics median	mean	top
1. Overall Homework	69%	50%	48%	91%
2. Pretest Regular Score	82%	60%	57%	88%
3. Pretest Bonus Score	0%	0%	10%	100%
4. Final Exam Regular Score	66%	50%	45%	90%
5. Final Exam Bonus Score	0%	0%	5%	100%
6. Overall Bonus (95% HW bon + 5% Final bon)	0%	3%	8%	41%
7. Overall Regular (70% HW reg + 30% Final reg)	68%	51%	45%	84%

II. CURRICULUM CONTENT

The attached **"MTRW F18 7A Group Awards Chart"** contains a list of the top performers in various categories.

Awards for some students (1st place score on page 1 and/or with a special recognition on page 2 of the 7A Group Awards Chart) may be picked up in Evans 714 during the office hours:

- MW 5:15 – 5:45pm, 6:15 – 7:45pm, January 21- February 20, 2019.

Any **unpicked HWs and exams** will be kept in front of Evans 714 until February 4, 2019.

Looking forward to having you again in future BMC/MTRW Programs,

Sincerely, December 21, 2018

Zvezdelina Stankova, Teaching Professor of Mathematics, University of California at Berkeley
Berkeley Math Circle, Math Taught the Right Way, Founder and Director

BERKELEY MATH CIRCLE

REPORT CARD

Student: Severus Snape

9A group at MTRW Fall 2018

I. HOMEWORK AND EXAM SCORES

I. 9A ALGEBRA, Instructor: Kelli Talaska	Student % of max	Class Statistics median	mean	top
1. Homework Regular	93%	50%	46%	93%
2. Homework Bonus	92%	3%	17%	92%
3. Pretest	0%	14%	18%	60%
4. Final Exam Regular	79%	60%	61%	94%
5. Final Exam Bonus	20%	0%	11%	80%
6. Improvement (Final Exam Regular – Pretest Regular)	79%	39%	43%	91%
7. Overall Regular (70% HW + 30% Final)	89%	56%	47%	89%
8. Overall Bonus (95% HW + 5% Final)	88%	3%	17%	88%
II. 9A GEOMETRY, Instructor: Norm Prokup	**Student % of max**	**Class Statistics median**	**mean**	**top**
1. Homework Regular	78%	44%	45%	92%
2. Homework Bonus	74%	1%	18%	84%
3. Pretest Regular	N/A	22%	27%	80%
4. Pretest Bonus	N/A	8%	13%	55%
5. Take-Home Test Regular	98%	83%	72%	100%
6. Take-Home Test Bonus	100%	55%	52%	100%
7. Final Test Regular	60%	52%	53%	96%
8. Final Test Bonus	47%	25%	33%	93%
9. Improvement (50% #4 + 5% #5 + 40% #6 + 5% #7 - 90% #3 Reg -10% #3 Bon)	77%	48%	46%	87%
10. Overall (50% #1 + 5% #2 + 0.9% #3 + 0.1% #4 + 20% #5 +2% #6 + 20% #7 + 2% #8)	N/A	36%	39%	86%
III. 9A PROOFS, Instructor: Zvezda Stankova	**Student % of max**	**Class Statistics median**	**mean**	**top**
1. Homework Regular	91%	30%	34%	91%
2. Homework Bonus	68%	2%	12%	80%
3. Pretest Regular	67%	62%	59%	100%
4. Pretest Bonus	0%	0%	14%	100%
5. Take-Home Final Test Regular	100%	83%	77%	100%
6. Take-Home Final Test Bonus	50%	0%	31%	100%
7. Final Exam Regular	82%	54%	54%	89%
8. Final Exam Bonus	85%	31%	36%	85%
9. Improvement (Final Test Reg – Pretest)	33%	42%	41%	75%
10. Overall Regular (50% HW + 10% Pretest + 10% Final Test + 30% Final Exam)	87%	44%	44%	87%
11. Overall Bonus (50% HW + 10% Pretest + 10% Final Test + 30% Final Exam)	64%	15%	20%	76%

IV. CURRICULUM CONTENT

The attached **"MTRW F18 9A Group Awards Chart"** contains a list of the top performers in various categories.
Awards for some students (marked with * on the Chart) may be picked up in Evans 714 during the office hours:

- MW 5:15 – 5:45pm, 6:15 – 7:45pm, January 21- February 20, 2019.

Any **unpicked HWs and exams** will be kept in front of Evans 714 until February 4, 2019.

Looking forward to having you again in future BMC/MTRW Programs,

Sincerely, ZStankova December 17, 2018

Zvezdelina Stankova, Teaching Professor of Mathematics, University of California at Berkeley
Berkeley Math Circle, Math Taught the Right Way, Founder and Director

16. Math Taught the Right Way

In addition to running the Tehiyah Day School math program on this textbook series for 3 years, we continued the weekly program *"Math Taught the Right Way"* (MTRW) for a 3rd year, extending the Berkeley Math Circle at UC Berkeley.

In fall 2018, MTRW had an increased enrollment of 135 students from 90 schools and 20 towns in the extended San Francisco Bay Area. The four program groups were selected mainly by age. As in the spirit of the Berkeley Math Circle, younger, more advanced students were promoted to higher groups. The fall groups were named 6A, 7A, 8A, and 9A.

	Taught by:
6A Group:	
• *6A Algebra*	Elena Blanter
• *6A Geometry*	Harry Main-Luu
• *6A Problem Solving*	Ellen Kulinsky
7A Group:	
• *7A Algebra*	Elena Blanter
• *7A Geometry*	Harry Main-Luu
• *7A Problem Solving*	Elysée W.-Egolf
8A Group:	
• *8A Algebra*	Kelli Talaska
• *8A Geometry*	Norm Prokup
• *8A Problem Solving*	Elysée W.-Egolf
9A Group:	
• *9A Algebra*	Kelli Talaska
• *9A Geometry*	Norm Prokup
• *9A Proofs*	Zvezdelina Stankova

In the 15 weekly 3-hour sessions in fall 2018,

- The *6A group* covered almost all of the 6A curriculum. *6A Problem Solving* concentrated on the *Divisibility* chapter and *Word Problems* from the 6A curriculum.
- *7A Algebra* covered *Proportions* and *Integer Expressions*. *7A Geometry* worked on *Polyhedral* and *Round Solids* in the 7A curriculum, and *Foundations of Geometry* in the 7B curriculum. *7A Problem Solving* worked its way through *Bulgarian Math Contests* for 6th-7th grades, learning problem-solving strategies and practicing ideas like solving through graphing and algebra, applying geometry theorems and formulas for difference and sum of squares formulas, and multiplying polynomials.
- *8A Algebra* covered *Equations Part II (Modeling and Applications)* from the 7B curriculum, and *Algebraic Inequalities* from the 8A curriculum. *8A Geometry* covered *Congruent Triangles, Part I* and *Parallelograms and Trapezoids* from the 8A curriculum. *8A Problem-Solving* was based on *Combinatorics* from the 8B curriculum and from *A Decade of the Berkeley Math Circle - the American Experience,* vol. I, edited by me and Tom Rike, and generalized to *Combinatorics of an n-dimensional cube analogue.*
- *9A Algebra* covered *Combinatorics* and *Equations and Roots* from the 8B curriculum, *Classical Probabilities* from the 9A curriculum, and practiced with *Rational Equations*. *9A Geometry* covered 2/3 of *Circle Geometry Parts I-II* from the 8B curriculum. *9A Proofs* covered *Rational Inequalities* from the 9A curriculum and *Inequalities, Part I* on AM-GM and other power mean inequalities, and the beginning of *Smoothing Inequalities* from *"A Decade of the Berkeley Math Circle - the American Experience,"* vol. I-II, edited by me and Tom Rike.

The Berkeley Math Circle Summer Program 2018 can be viewed as a third component of the MTRW Program, which completes the 6th-9th grade curriculum, as well as some math circle topics. The faculty who taught at the Summer Program were:

- Chris Overton, Ph.D., Ayasdi
- Ellen Kulinsky, UC Berkeley
- Elysée W.-Egolf, B.A. UCB, Bentley High
- Harry Main-Luu, BMC asst, UC Berkeley
- Joshua Zucker, M.S. Stanford
- Kelli Talaska, Ph.D., UC Berkeley
- Laura Pierson, BMC alumna, Harvard
- Maia Averett, Ph.D., Mills College
- Matyas Sustik, Ph.D., WalmartLabs
- Norm Prokup, Ph.D., College Prep, Oakland
- Quan Lam, Ph.D., MBA, UC President's Office
- Zvezdelina Stankova, Ph.D., UC Berkeley