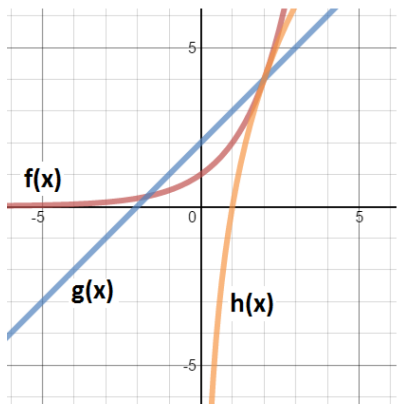
MATH WRANGLE 2016 SOLUTIONS

1 $(a,b,c,d,y) = \left(2,1,2,\sqrt[4]{2},2\right)$ **will work.**

- $f(x) = 2^x$ so $f(2) = 2^2 = 4$
- $g(x) = x+2$ so $g(2) = 2+2 = 4$
- $h(x) = \log_{\sqrt[4]{2}}(x)$ so $h(2) = \log_{2^{\frac{1}{4}}}(2) = 4$ since $\left(2^{\frac{1}{4}}\right)^4 = 2$.



In fact there are an infinity of solutions which might be found as follows: take any $a > 1$ and any line of positive slope which must intersect the exponential in the first quadrant. We can then thread a logarithm through this point of intersection.

2 **The answer is 2016**. We will prove more generally that given any two squares in a plane, the midpoints of the line segments connecting corresponding vertices of the two squares is itself another square.

For an intuitive proof, imagine a situation similar to the diagram, where a small square lies inside a larger square. One can visualize this as a view from above of a three-dimensional shape where every cross-section parallel to the ground is a square, with the large square at the base and the small square as the roof. Then the midpoints in question are just the vertices of the layer halfway up; a square.

A more rigorous approach uses complex numbers. Recall that multiplication by i corresponds to counter-clockwise rotation by 90°. Since $ABCD$ and $PQRS$ are squares, it must be the case that $(D-C)i = B-C$ and $(S-R)i = Q-R$. To verify that X,Y,Z are vertices of a square, we merely need to check that $(X-Y)i = Z-Y$.

Since X,Y,Z are respectively, $\frac{1}{2}(D+S), \frac{1}{2}(C+R), \frac{1}{2}(B+Q)$, we have

$$(X-Y)i = \frac{1}{2}(D-C+S-R)i = \frac{1}{2}(B-C) + \frac{1}{2}(Q-R) = \frac{1}{2}(B-C+Q-R) = Z-Y.$$

Notice that the information in the original problem regarding collinearity and the degree angle measures of 15° and 120° are red herrings.

3 **The answer is 5**. Let the smaller of the two primes be congruent to n mod 9. If $n = 1$, the larger one is congruent to 3 modulo 9, which is impossible, as no number congruent to 3 mod 9 can be prime (other than 3). Likewise, n cannot equal 3, 4, 6, or 7. The only possible choices are $n = 2,5,8$, in which case $n+2$ is congruent to $4,7,1$, respectively, and in all three cases, the product will be congruent to 8 modulo 9. We can determine the unknown digit Z using the well-known method of "casting out nines:" a number is congruent (mod 9) to the sum of its digits. The digit sum of $16926Z8244483$ is congruent to $Z+3 \pmod 9$, hence $Z = 5$. By the way, $1692658244483 = (1301021)(1301023)$ and 1301021 is congruent to 8 modulo 9.

4 **The answer is 9601**. Finding the last 4 digits is equivalent to computing the number modulo 10,000. It is easy to observe $7^4 = 2401$. Using the binomial theorem, note that $(a+1)^5 = p(a)+5a+1$, where $p(a)$ is a polynomial with integer coefficients whose lowest-degree term is degree 2. With $a = 2400$, we see that a^k for $k \geq 2$ will end in 0000. Therefore, like $5(2400)+1 = 12001$ we have that $(a+1)^5 = 2401^5$ will end in 2001. Using the same idea, we see that $2001^5 \equiv 5(2000)+1 = 10001 \equiv 1 \pmod{10000}$; i.e., $7^{100} \equiv 1 \pmod{10000}$, and likewise $7^{2000} = (7^{100})^{20} \equiv 1 \pmod{10000}$. So we just need to determine $7^{16} \pmod{10000}$.

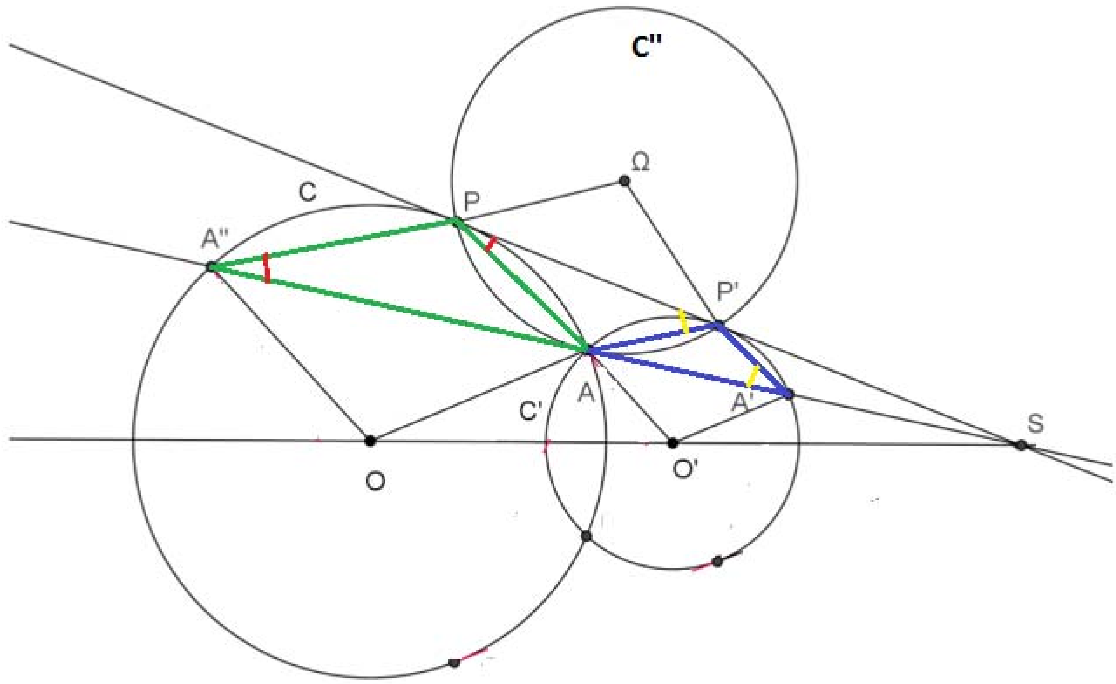
Since $7^{16} = 2401^4$ we again use the binomial theorem to compute $(a+1)^4 = q(a)+4a+1$, with $q(a)$ a polynomial with integer coefficients whose lowest-degree term is degree 2. With $a = 2400$ Thus we have that 2401^4 like $4(2400)+1 = 9601$ ends with the digits 9601.

5 Let S be the external center of similitude for circles C, C', and let A", A' be the second points of intersection of line SA with circles C, C' respectively (see diagram). This means that the radii of the circles C and C' are proportional to the distance between their centers (O and O') and S.

In the same way then clearly triangles APA'' (GREEN) and $A'P'A$ (DARK BLUE) are homothetic with center S of homothecy. (In particular, they are similar.) Also, $\angle PA''A = \angle APP'$ (marked in RED) since each is inscribed in the same arc $\overset{\frown}{AP}$ of circle C. Likewise, $\angle P'A'A = \angle AP'P$ YELLOW $= \frac{1}{2}$arc $\overset{\frown}{AP'}$ in circle C'. It follows that triangles $\triangle A''PA$ GREEN , $\triangle PAP'$ GREEN-BLUE-BLACK , and $\triangle AP'A'$ BLUE are all similar.

Now we note that the three circumcenters O, O', Ω of these triangles, considered as parts of the triangles, are corresponding points. Therefore quadrilaterals $A''PAO, PAP'\Omega$, and $AP'A'O'$ are also all similar. From these three quadrilaterals we have: $\angle OAP = \angle \Omega P'A$ and $\angle O'AP' = \angle \Omega PA$. Therefore

$$\angle P\Omega P' = 360^\circ - \angle \Omega PA - \angle PAP' - \angle \Omega P'A = 360^\circ - \angle O'AP' - \angle PAP' - \angle OAP = \angle OAO'$$



Where the last equality comes from going all the way around the intersection point A. That is, the angle subtended by the arc $\overset{\frown}{PP'}$ from the center Ω of the circle C'' circumscribing triangle $\triangle APP'$ is equal to the angle between the radii of circles C, C' drawn to their point of intersection. It is not hard to see that this angle is equal to the angle between the tangents to these circles at their point of their intersection, which is the angle between the two circles themselves.

6 **The answer is "A."** Let s_n denote the sequence of Vs and As after folding the paper n times. It is easy to verify that

- $s_1 = V$
- $s_2 = VVA$
- $s_3 = VVAVVAA$,

and that, in general s_{n+1} is formed by concatenating s_n, a single "V", and a "back flip" of s_n, where back flip means "write the sequence in reverse and change each symbol to its opposite." (For example, the back flip of $AAVVVA$ would be $VAAAVV$.)

There are several consequences of this recursion principle:

1. s_n has $2^n - 1$ symbols;
2. The 2^kth term of any s_n is always "V."

3. If $n < m$ then s_n is the initial string of s_m.

Therefore, if we want to know the 2016th term of s_{2016}, we only need to look at the 2016th term of s_{11} which has $2^{11} - 1 = 2047$ symbols. The 2016th term of s_{11} is the 32nd term *from the right* of s_{11}. Since $s_{11}$is the concatenation of s_{10}, a "V", and the *back flip* of s_{10}, this term will be the opposite of the 32nd term of s_{10} *from the left*, which is "V," using rule #2 (since $32 = 2^5$). Thus the term we seek is the opposite of "V," which is "A."

7 **The answer is 4**. Observe that N is divisible by both 2 and 3 and thus is a multiple of 6. Hence we can write $N = 6u$ for some positive integer u. Suppose we remove from S the consecutive integers $a-1, a, a+1$, where $2 \le a \le 6u-1$. Then we have removed a sum of $3a$ from S. Recall that the sum of the first N integers is $\frac{N(N+1}{2}$ and thus the average of what remains is $\frac{\frac{6u(6u+1)}{2}-3a}{6u-3} = \frac{3u(6u+1)-3a}{6u-3} = \frac{18u^2+3u-3a}{6u-3} = \frac{6u^2+u-a}{2u-1}$ and this must be an integer.

$$= \frac{6u^2 - 3u + 4u - a}{2u-1} = \frac{3u(2u-1)}{2u-1} + \frac{4u-a}{2u-1} = 3u + \frac{4u-a}{2u-1}$$

So the problem reduces to finding those values of a for which $\frac{4u-a}{2u-1}$ is an integer. Since $2(2u-1) \ge 4u - a$ the only possible multiples are $-1, 0, 1$ and 2. When $4u - a = 2(2u-1) = 4u-2$ we have $a = 2$. When $4u - a = 2u - 1$ we have $a = 2u$. When $4u - a = 1 - 2u$ we have $a = 6u - 1$ and finally, when $4u - a = 0$ we have $a = 4u$.

8 We first note that because of the circle sizes the respective slow angular speeds of Ark, Bark, and Spark are $\pi, \frac{\pi}{2}$, and $\frac{\pi}{3}$ while their respective fast angular speeds are $2\pi, \pi$, and $\frac{2\pi}{3}$ all in radians per second. We record the respective locations of each bug according to their radian angle at each non-negative integer value of t as follows:

Time t Seconds	**Ark**	**Bark**	**Spark**
0	0	0	0
1	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$
2	$\frac{\pi}{2}$ and **1 Lap**	π	$\frac{2\pi}{3}$
3	0 and **2 Laps**	0 and **1 Lap**	π
4	π and **2 Laps**	$\frac{\pi}{2}$ and **1 Lap**	$\frac{5\pi}{3}$
5	$\frac{\pi}{2}$ and **3 Laps**	π and **1 Lap**	$\frac{\pi}{6}$ and **1 Lap**
6	0 and **4 Laps**	0 and **2 Laps**	$\frac{\pi}{2}$ and **1 Lap**
7	π and **4 Laps**	$\frac{\pi}{2}$ and **2 Laps**	$\frac{5\pi}{6}$ and **1 Lap**
8	$\frac{\pi}{2}$ and **5 Laps**	π and **2 Laps**	$\frac{4\pi}{3}$ and **1 Lap**
9	0 and **6 Laps**	0 and **3 Laps**	0 and **2 Laps**

The last entry of $t = 9$ gives us our solution. For a correct solution, however, it is important to note that a smaller time t does not occur between these integer values. However, this possibility can be dismissed because even with respect to Ark and Bark the only coincidences of location occur at the given integer values of $t = 3, 6$, and 9 and these are always the result of Ark being about to lap Bark. At other times Ark does not pass Bark's location.