Minimum Volume between a Surface and its Tangent Plane

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Math Circles: It's Circular: Conjecture, Compute, Iterate

This is a an interesting circle for high school students who have already had calculus. It can be done for the area between a the graph of a function of 1 variable and its tangent line or for the volume between a function of 2 variables and its tangent plane. In the interest of time we will do the 1 variable problem here.

Everyone get out some paper and pick a function which is concave up or down on the interval $[0,1]$. Here are some possibilities:

$$f(x) = x^2 \qquad f(x) = e^{-x} \qquad f(x) = \cos x \qquad f(x) = \ln(1+x) \qquad f(x) = x^4$$

I will use

$$f(x) = e^x$$

1. Find the tangent line at $x = \dfrac{1}{4}$: $\quad f\left(\dfrac{1}{4}\right) = e^{1/4} \quad f'(x) = e^x \quad f'\left(\dfrac{1}{4}\right) = e^{1/4}$

$$y = f_{\tan}(x) = e^{1/4} + e^{1/4}\left(x - \frac{1}{4}\right)$$

2. Find the area between $y = f(x)$ and its tangent line at $x = \dfrac{1}{4}$:

$$\begin{aligned} A &= \int_0^1 f(x) - f_{\tan}(x)\,dx = \int_0^1 e^x - e^{1/4} - e^{1/4}\left(x - \frac{1}{4}\right)\,dx \\ &= e - 1 - \frac{5}{4}e^{1/4} \end{aligned}$$

3. Find the tangent line at $x = p$: $\quad f(p) = e^p \quad f'(x) = e^x \quad f'(p) = e^p$

$$y = f_{\tan}(x) = e^p + e^p(x - p)$$

4. Find the area between $y = f(x)$ and its tangent line at $x = p$:

$$A(p) = \int_0^1 f(x) - f_{\tan}(x)\,dx = \int_0^1 e^x - e^p - e^p(x - p)\,dx$$
$$= e - 1 - e^p - e^p\left(\frac{1}{2} - p\right)$$

5. Find the point of tangency $x = p$ for which the area is a minimum. First find the critical points:

$$A'(p) = -e^p - e^p\left(\frac{1}{2} - p\right) - e^p(-1) = -e^p\left(\frac{1}{2} - p\right) = 0$$

$$\Rightarrow \quad p = \frac{1}{2}$$

6. Then check it is a minimum:

$$A''(p) = -e^p\left(\frac{1}{2} - p\right) - e^p(-1) \qquad \Rightarrow \qquad A''\left(\frac{1}{2}\right) = e^p > 0$$

Or

$$A\,(0) = e - 1 - 1 - \frac{1}{2} = e - 2.5 \approx .22$$
$$A\left(\frac{1}{2}\right) = e - 1 - \sqrt{e} \approx .07$$
$$A\,(1) = e - 1 - e + \frac{1}{2}e = \frac{e}{2} - 1 \approx .36$$

7. Prove the point of tangency $x = p$ for which the area is a minimum is $p = \dfrac{1}{2}$ no matter what concave up or concave down function you pick!

8. Let $f(x) = g(x)$ be an arbitrary function which is concave up on $[0, 1]$. This means $f''(x) = g''(x) > 0$ for all $x \in [0, 1]$.

9. The tangent line at $x = p$ is:

$$y = f_{\tan}(x) = g(p) + g'(p)(x - p)$$

10. The area between $y = f(x)$ and its tangent line at $x = p$ is:

$$\begin{aligned} A(p) &= \int_0^1 f(x) - f_{\tan}(x)\, dx = \int_0^1 g(x) - g(p) - g'(p)(x - p)\, dx \\ &= \int_0^1 g(x)\, dx - g(p) - g'(p)\left(\frac{1}{2} - p\right) \end{aligned}$$

11. Finally, find the point of tangency $x = p$ for which the area is critical:

$$A'(p) = -g'(p) - g''(p)\left(\frac{1}{2} - p\right) - g'(p)(-1) = -g''(p)\left(\frac{1}{2} - p\right) = 0$$

$$\Rightarrow \quad p = \frac{1}{2}$$

since $g''(p) > 0$.

12. And check it is a minimum:

$$A''(p) = -g'''(p)\left(\frac{1}{2} - p\right) - g''(p)(-1) \qquad \Rightarrow \quad A''\left(\frac{1}{2}\right) = g''(p) > 0$$

again since $g''(p) > 0$. Or

$$
\begin{aligned}
A\left(0\right) &= \int_0^1 g(x)\,dx - g(0) - \frac{1}{2}g'(0) \\
A\left(\frac{1}{2}\right) &= \int_0^1 g(x)\,dx - g\left(\frac{1}{2}\right) \\
A\left(1\right) &= \int_0^1 g(x)\,dx - g(1) + \frac{1}{2}g'(1)
\end{aligned}
$$

13. Similar result in Calc 3:
The volume between a surface and its tangent plane over the square $[0, 1] \times [0, 1]$ is a minimum at $(p, q) = \left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ for any graph $z = f(x, y)$ which is concave up or concave down on the region!

14. More generally, for any region (not a square) the minimum will occur at the centroid.