Counting Cube Colorings

Notes For Circle Leaders

I saw this topic presented to a group of students at the Boston Math Circle at Northeastern University. The presenter was a professor at a college in the greater Boston area, but I do not recall her name. Since then, I have shared this topic with many groups of students. I love that it requires very little prior knowledge of high school mathematics, though references to properties of operations and function composition provide touch stones for students who have had some algebra. I also love the fact that students can be introduced to group theory, and then can use it to solve a wide range of challenging problems. When presented as a 90 minute talk, here is the pacing that I have used. When I have used this with my Math Circle students, I give the students much more time on each part and we tend to go farther afield exploring side questions that they bring up. After playing with this question, students could move on to creating their own similar problems, including in situations where the acting group is a permutation group.

10 minutes: As students arrive, invite them to use pipe cleaners (only one color) and pony beads (in two colors) to build cubes and squares. They should put exactly one pony bead at each vertex.

10 minutes: How many different colorings of a cube are possible? Students think about 2^8 as a possible solution, and then realize that this over-counts by quite a bit because of the symmetries of a cube. Discuss the definition of a symmetry, and identify symmetries in the cube and in the square.

10 minutes: Introduce the concept of a group in general, and the symmetry group of a square in particular.

15 minutes: Fill in the Orbits and Stabilizers chart for the group of symmetries of a square.

15 minutes: Notice and prove the Orbit-Stabilizer theorem and Burnside's Lemma.

15 minutes: Use Burnside's Lemma to find the number of orbits for the cube.

15 minutes: Extensions, conclusions.

Counting Cube Colorings

Identity: 1, stabilizes 256 colorings

1/2 turn face rotations: 3, each stabilizes 16 colorings

1/4 turn face rotations: 6, each stabilizes 4 colorings

1/2 turn edge rotations: 6, each stabilizes 16 colorings

1/3 turn vertex rotations: 8, each stabilizes 16 colorings

Total colorings: $2^8 = 256$

Total symmetry group elements: $1 + 3 + 6 + 6 + 8 = 8 \cdot 3 = 6 \cdot 4 = 12 \cdot 2 = 24$

Total stabilizers: $1 \cdot 256 + 3 \cdot 16 + 6 \cdot 4 + 6 \cdot 16 + 8 \cdot 16 = 552$

Total orbits: $552/24 = 23$

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Orbits and Stabilizers for the Group of Square Symmetries

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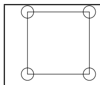
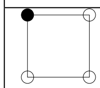
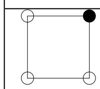
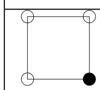
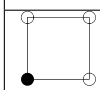
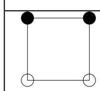
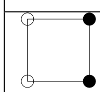
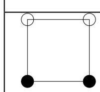
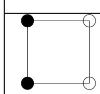
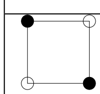
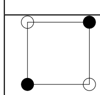
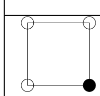
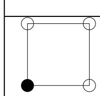
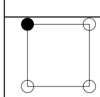
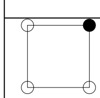
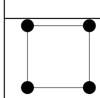
	X	X	X	X	X	X	X	X
	X							X
	X						X	
	X							X
	X						X	
	X					X		
	X				X			
	X					X		
	X				X			
	X		X				X	X
	X		X				X	X
	X							X
	X						X	
	X							X
	X						X	
	X	X	X	X	X	X	X	X

Counting Cube Colorings

1. How many different ways are there to color the vertices of a cube using two colors? This question is easier to answer if the cube is not able to move. What would the answer be in that case?

2. If the cube can be moved, why is this question more difficult to answer?

3. Let's consider a simpler version of this problem. How many different ways are there to color the vertices of a square using two colors? It will be helpful to investigate the symmetries of a square, and the ways the vertices of the square can be colored if the square does not move.

4. In the top row of the chart, list all of the symmetries of a square (leave the upper left corner blank). A symmetry is a motion that rearranges the locations of the vertices and edges while returning the square to the same position it was in to begin with. The simplest symmetry is the "do nothing" motion or "identity". Be sure to create notation that makes it clear how each symmetry rearranges the vertices.

5. The symmetries of a square form a "group". A group consists of a set of objects and an operation that combines them subject to a few rules. In this case, the objects are the different motions of the square. The operation combining them is composition – make the first motion first followed by the second motion. Any two motions is equivalent to another single motion. A group must have four properties. The first is closure – whenever any two objects are combined, they must form a new object that is included in the group. The second is the existence of an identity element. When this element is combined with any other element, the result is the other element (the identity element does not change the other element). The third property is the existence of inverse elements – each element in the group must have an inverse in the group so that when the two of them are combined they result in the identity element. The last property is associativity – rearranging parentheses should not effect the outcome when three elements are combined. Commutativity (rearranging the order of two elements without changing the result) is not a required property for a group. In the case of the symmetry group for a square, some symmetries are not commutative. Can you find two elements that do not commute?

6. Along the left edge of the chart, draw pictures showing all of the different ways to color the vertices of a square using two colors if the square does not move. Try to group squares that are the same except for position next to each other in the chart.

7. Squares that are the same as each other but which are in a different position form an "orbit" under the action of the symmetry group. Use lines on the side of the chart to group the colored squares drawn on the left edge into orbits.

8. We say that a symmetry "stabilizes" a colored square in a given position if the action of that symmetry does not change the appearance of the colored square. For each row in your chart, mark the symmetries that stabilize the pictured square with an X.

9. What do you notice about the numbers of X's (stabilizers) in the various orbits? This idea is called the Orbit-Stabilizer Theorem.

10. What is the total number of stabilizers in the chart? What is the number of orbits (different ways to color a square allowing for motions)? What do you notice about these numbers? This idea is usually referred to as Burnside's lemma or the Cauchy–Frobenius lemma.

11. Try using Burnside's lemma to find the number of ways to color the vertices of a cube. Begin by counting the number of symmetries that a cube has. Remember to include the identity symmetry in your list. What is the total number of symmetries?

12. How many different ways are there to color a cube if it cannot be moved?

13. For each symmetry, think about how many different colored cubes would be stabilized by that motion. What is the total number of stabilizers in the chart?

14. Use Burnside's lemma to find the number of orbits among the colored cubes. How many different ways are there to color the vertices of a cube if it can be moved?

Orbits and Stabilizers of the Group of Symmetries of a Square
