Permutations, Folding and Dragons

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A puzzle:

Insert the numbers 1 through 10 and respect the inequalities.

$$\boxed{10} > \boxed{1} < \boxed{9} > \boxed{8} > \boxed{2} < \boxed{7} > \boxed{3} < \boxed{4} < \boxed{6} > \boxed{5}$$

This puzzle has 21,319 solutions.

Can puzzles like these always be solved?

EXTREME CASES:

Least number of solutions: $\square > \square > \square > \square > \square > \square > \square > \square > \square > \square$

Most number of solutions: $\square > \square < \square > \square < \square > \square < \square > \square < \square > \square$

Average number of solutions?

N symbols

N-1 inequalities

$\sim \;=\; < \text{ or } >$

$$\square \sim \square \sim \square \sim \square \sim \square \sim \square \sim \square \sim \square \sim \square \sim \square$$

Answers

$N!$ **permutations**

Questions

2^{N-1} **arrangements of inequalities**

On average a puzzle has $\dfrac{N!}{2^{N-1}}$ solutions.

On average a puzzle has $\dfrac{N!}{2^{N-1}}$ solutions.

N	Ave Number
1	1
2	1
3	1.5
4	3
5	7.5
6	22.5
7	78.75
8	336
9	1417.5
10	7087.5

$\dfrac{N!}{2^{N-1}}$ is an integer

if, and only if,

N is a power of two.

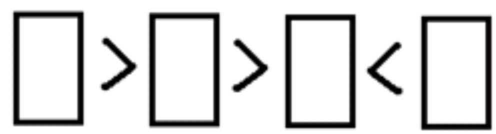
ALEX SMITH:

When N is a power of two, is this average answer actually realised?

e.g. For N = 4, is there a puzzle with precisely 4!/8 = 3 solutions?

For N = 8, is there a puzzle with precisely 8!/128 = 336 solutions?

In these cases ... YES!



Has 3 solutions

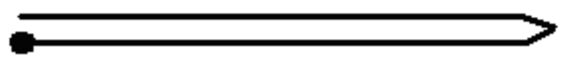
What are the three solutions?

4 > 3 > 1 < 2
4 > 2 > 1 < 3
3 > 2 > 1 < 4

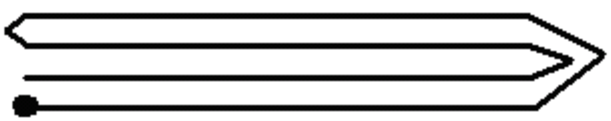
Has 336 solutions

COMPLETE CHANGE OF TOPIC … **FOLDING PAPER**

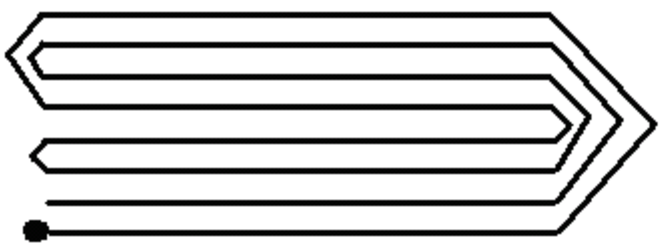
one fold



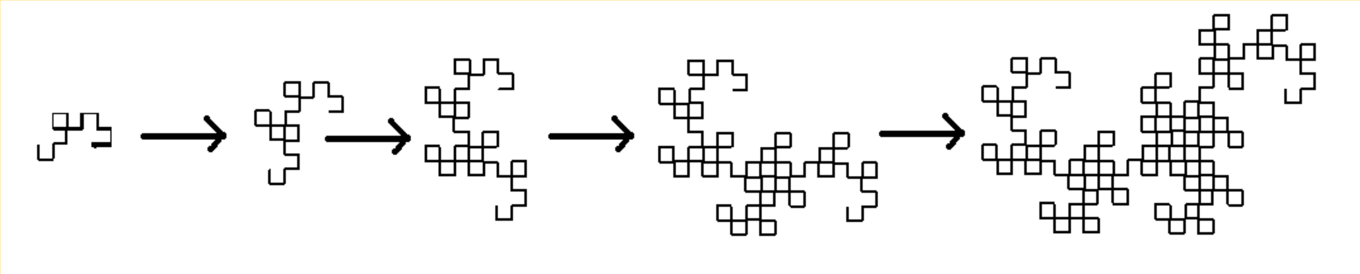
two folds

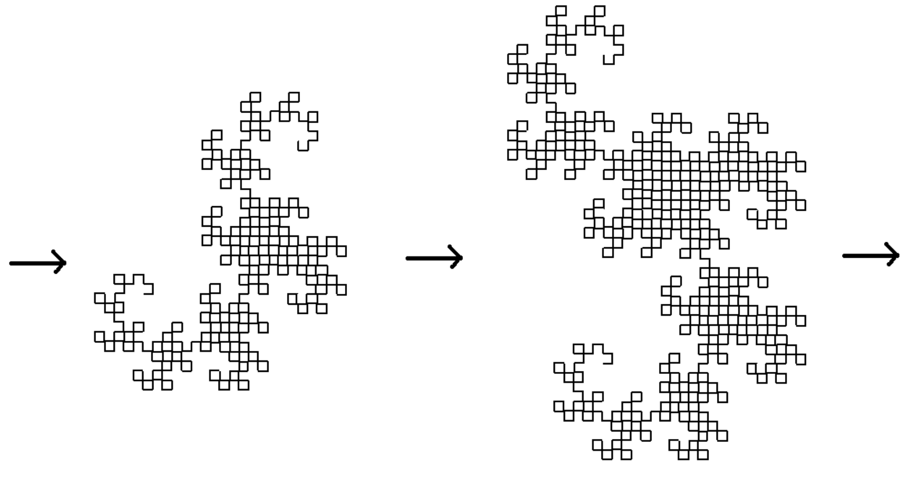


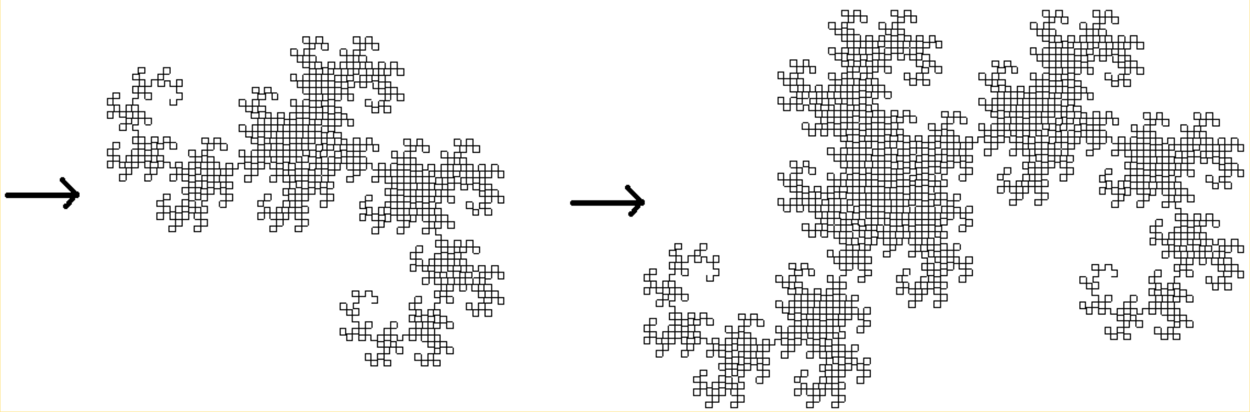
three folds

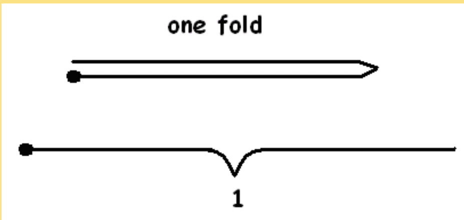


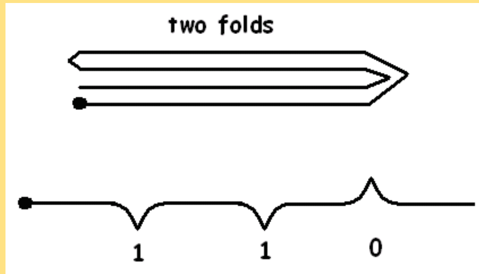
And so on …

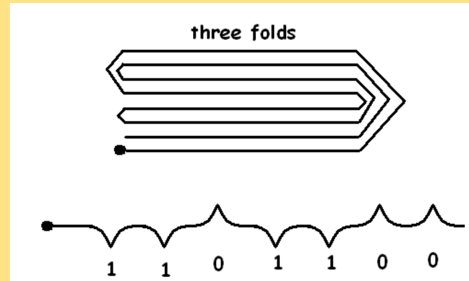








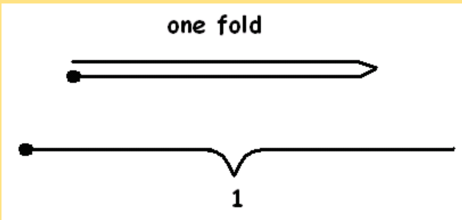




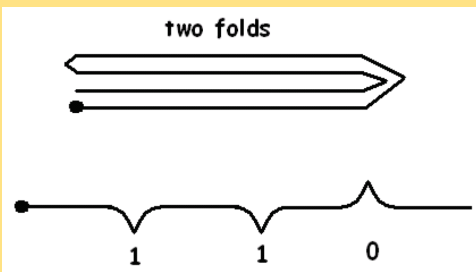
The sequences of valleys and mountain creases are curious!

1
110
1101100
110110011100100
1101100111001001110110001100100
etc

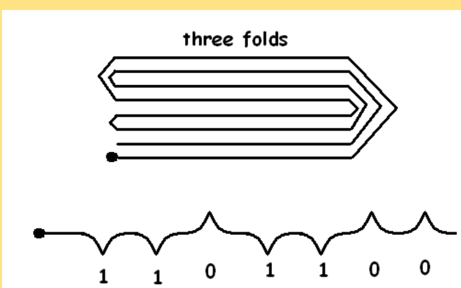
Alex's Discovery …



= □>□ 1 solution



= □>□>□<□ 3 solutions



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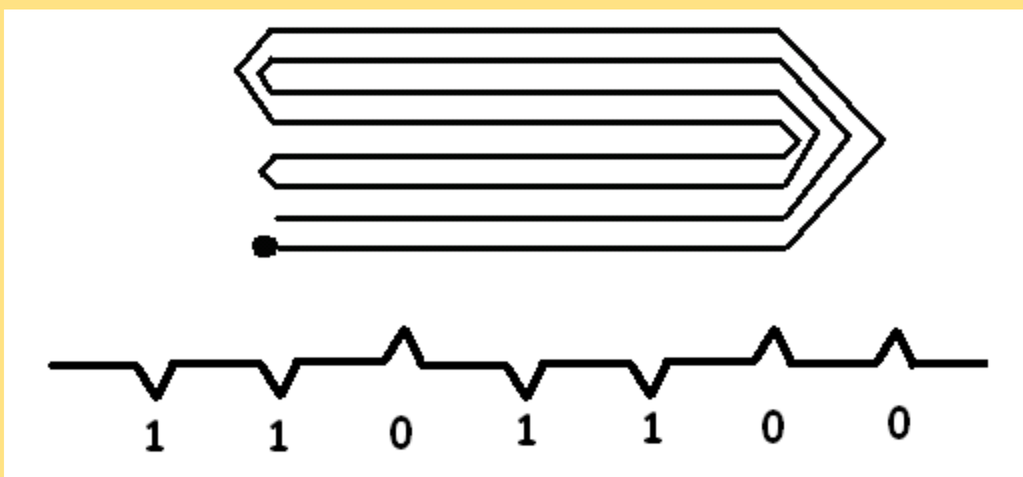
336 solutions

These are the patterns from always folding right to left.

MIX IT UP!



A.SMITH THEOREM:

Let $N = 2^k$.

If one folds a strip of paper in half k times, in any manner of one's choosing, then the mountain and valley creases give an inequality problem with precisely the average number of solutions: $\dfrac{N!}{2^{N-1}}$.

THE A.SMITH CONJECTURE:

If an inequality has the average number of solutions, then it must be an inequality from folding.

All is discussed and proved

(except the A. Smith Conjecture is left an exercise for the reader)

in a new MAA book!

MATHEMATICS GALORE!
The First Five Years of the St. Mark's Institute of Mathematics

WooHoo!!