Visualize the Two Conjugate Complex Roots for Quadratic Equations

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Introduction

Quadratic equations and quadratic functions are two basic must-have topics in high school algebra and in freshmen college algebra.

Quadratic Equation
$ax^2 + bx + c = 0, (a \neq 0)$

Its solutions (roots) have three cases.

Case 1. The discriminant $D = b^2 - 4ac > 0$, there are two real solutions:

$$x_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

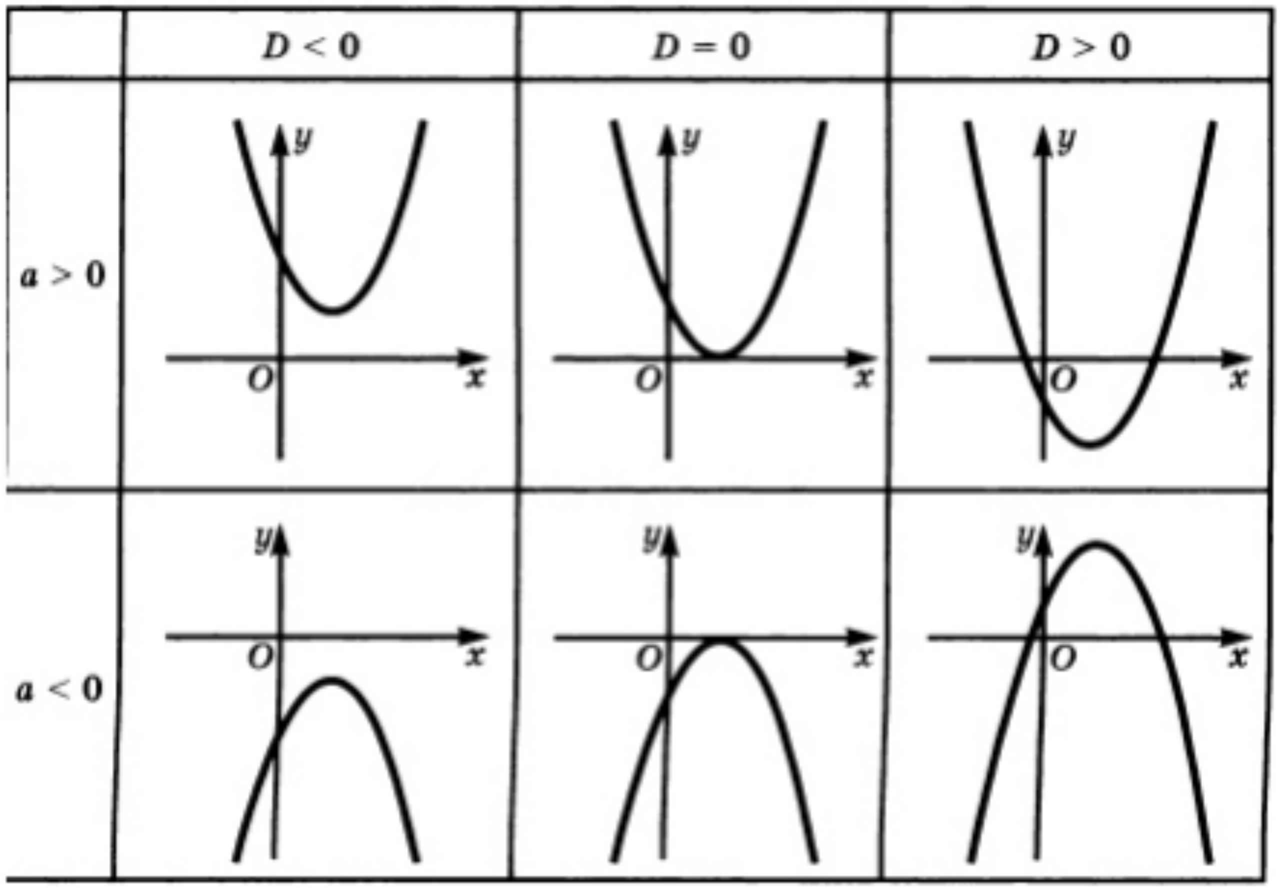
Case 2. The discriminant $D = b^2 - 4ac = 0$, there is one repeated real solution and $x = -\dfrac{b}{2a}$.

Case 3. The discriminant $D = b^2 - 4ac < 0$, there are no real solutions, but there are two conjugate complex solutions:

$$x_1 = \frac{-b}{2a} + \frac{\sqrt{|b^2 - 4ac|}}{2a} i \quad \text{and} \quad x_2 = \frac{-b}{2a} - \frac{\sqrt{|b^2 - 4ac|}}{2a} i$$

where $i = \sqrt{-1}$ is the imaginary unit.

$$f(x) = ax^2 + bx + c, \ (a \neq 0)$$



The graphs help students to see where the real solutions are.

- The last case, the discriminant $D = b^2 - 4ac < 0$, the graph just shows there is no x-intercept.
  - ***Where are the two conjugate complex solutions?***
  - ***Can the two conjugate complex solutions be found by identifying some points, just like when identifying the x-intercepts in the first two cases.***

A Geometric Way to Find the Complex Solutions

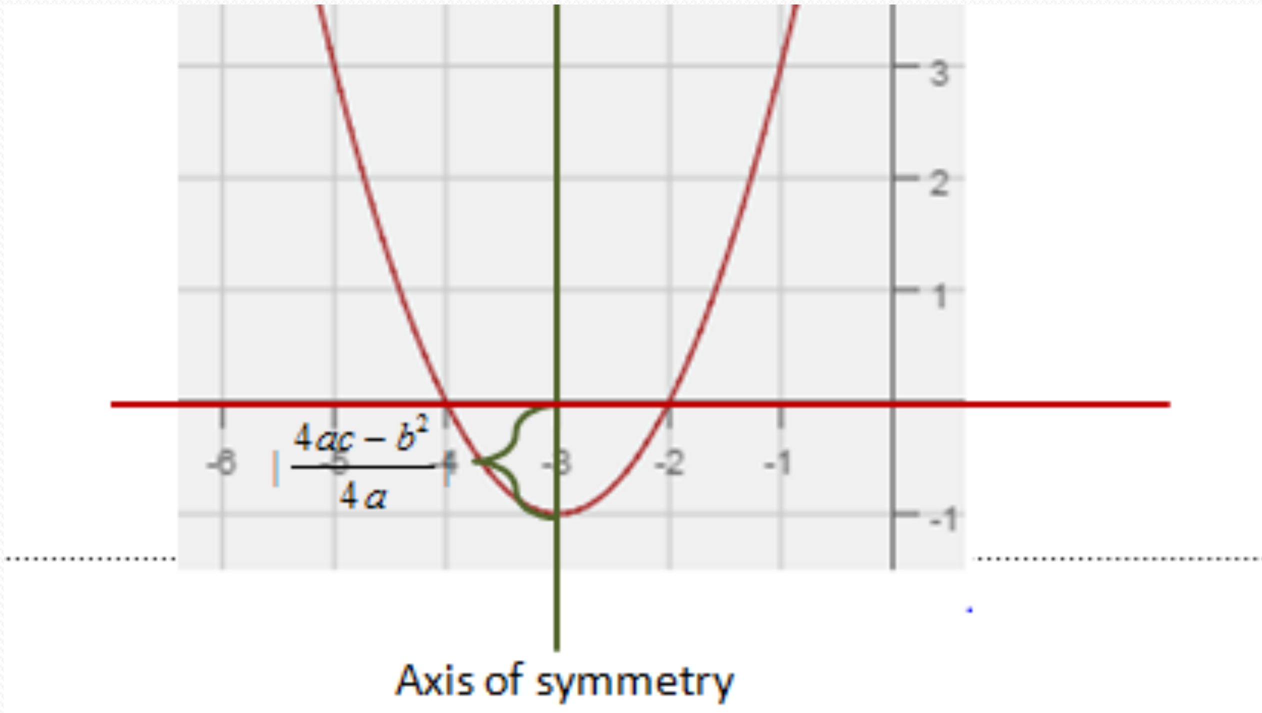
In the case $D = b^2 - 4ac > 0$, we try another way to identify the two real solutions.

Recall

For a quadratic function $f(x) = ax^2 + bx + c$, $(a>0)$,

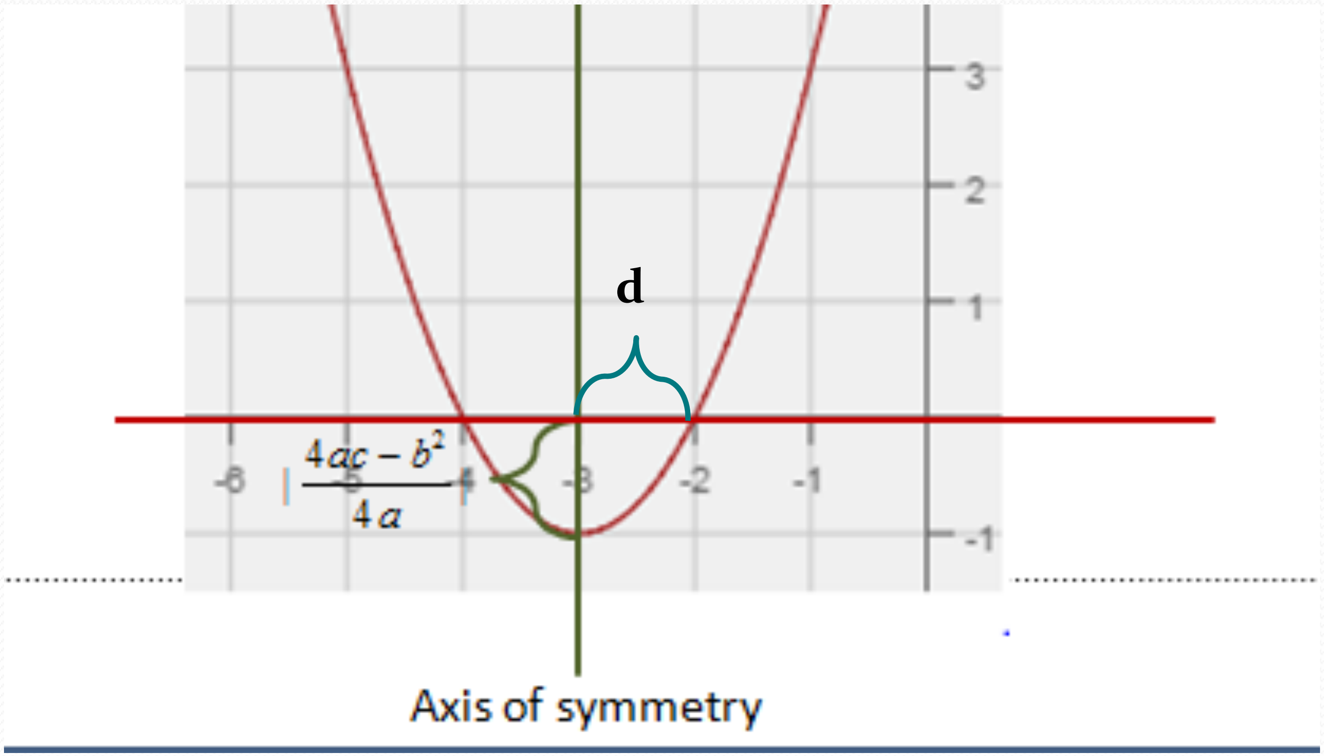
- **Its graph is a parabola**
- **The vertex is** $(-\frac{b}{2a}, \frac{4ac-b^2}{4a})$
- **The axis of symmetry** $x = -\frac{b}{2a}$.

Example: $f(x) = x^2 + 6x + 8$



- Vertex (-3, -1) and the axis of symmetry x = -3.
- Horizontal line intersects with the parabola at two points $(-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a},0)$ (-2, 0) and $(-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a},0)$ (-4,0).

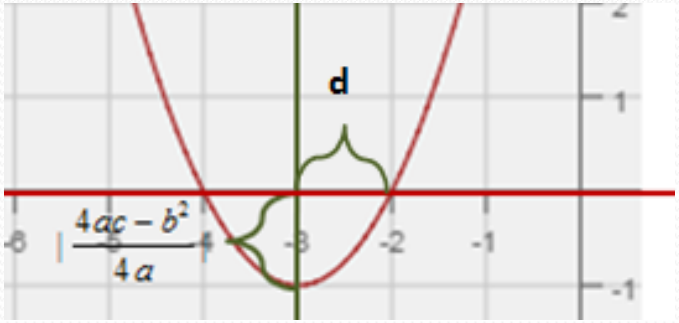
Example: $f(x) = x^2 + 6x + 8$



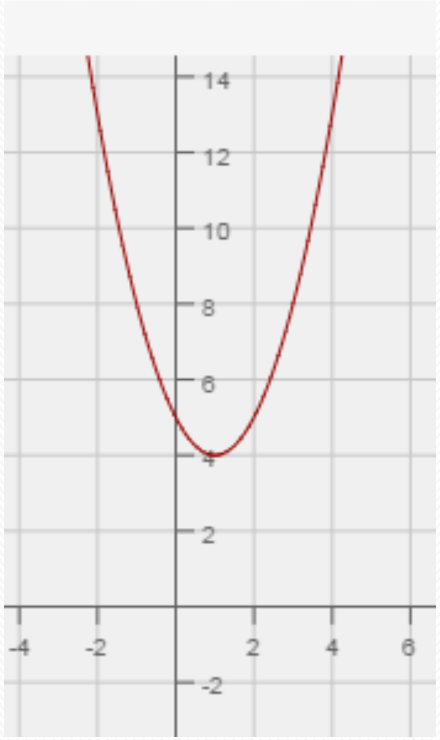
- The distance from any of the intersection points to the axis of symmetry is $d = \frac{\sqrt{b^2-4ac}}{2a} = \frac{\sqrt{|D|}}{2a}$, (d = 1) .

Hence we can view the solutions $-\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = -\frac{b}{2a} \pm d$ as the composition of two parts.

- First part is the location of the axis of symmetry
- Second Part is d, the distance from the intersection point to the axis of symmetry



$f(x)=ax^2 + bx + c$, $(a>0)$ when the discriminant $D = b^2-4ac <0$



$x^2 - 2x + 5 = 0$

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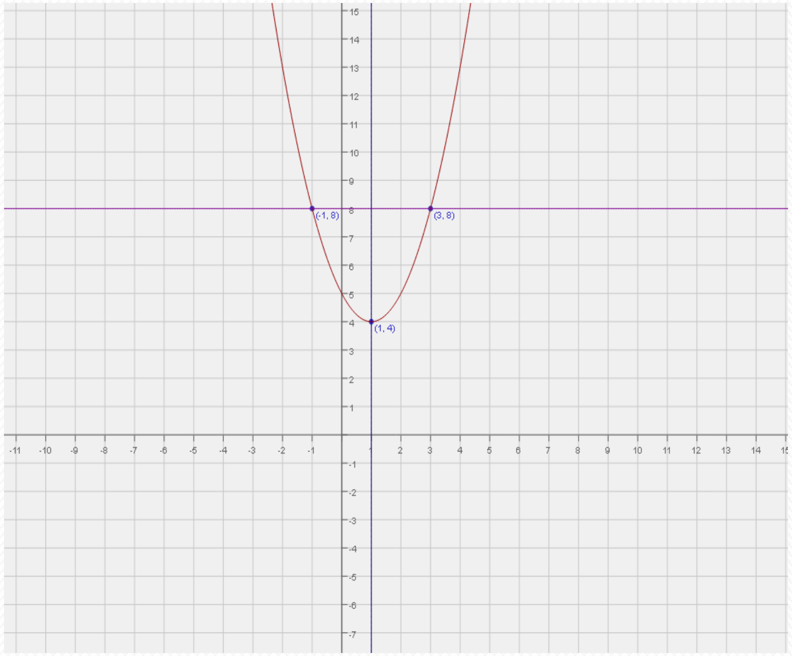


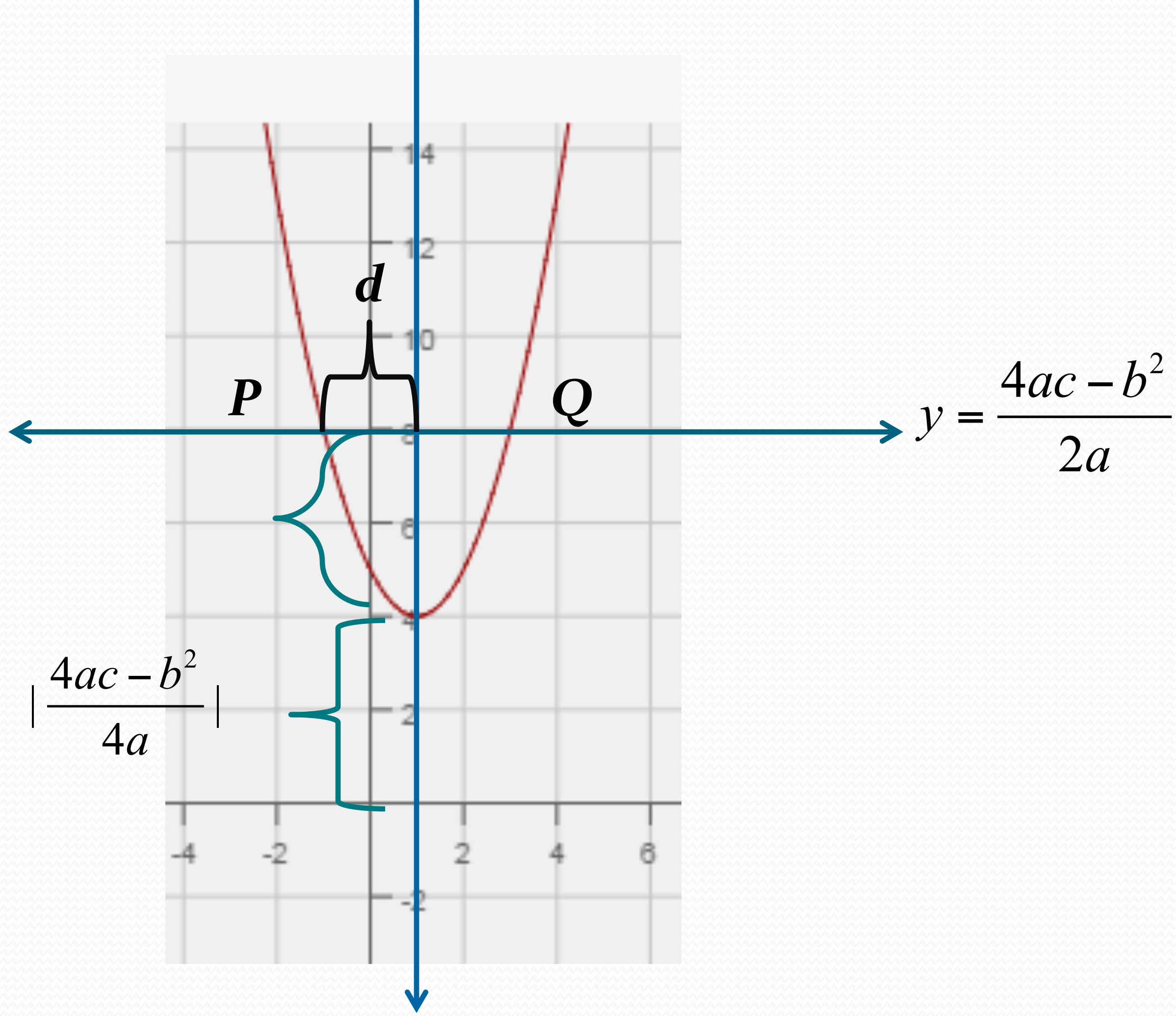
The vertex is (1, 4)

Axis of symmetry is x = 1

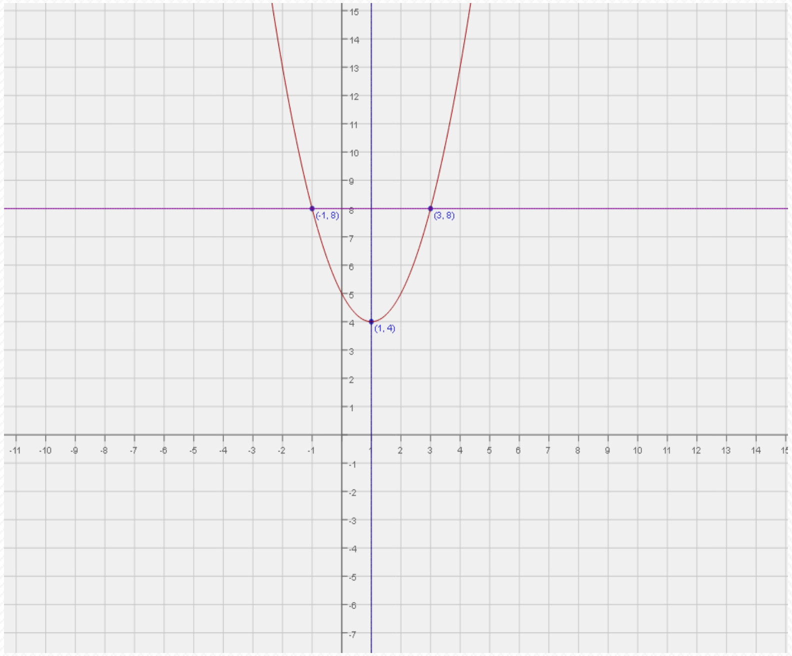
From the vertex go up 4 units along the axis of symmetry.

Then draw a horizontal line.

$X^2 - 2x + 5 = 0$



The horizontal line is cut through the parabola at two point ***P*** and ***Q***. Let ***d*** denote the distance from the point ***P*** to the axis of symmetry.



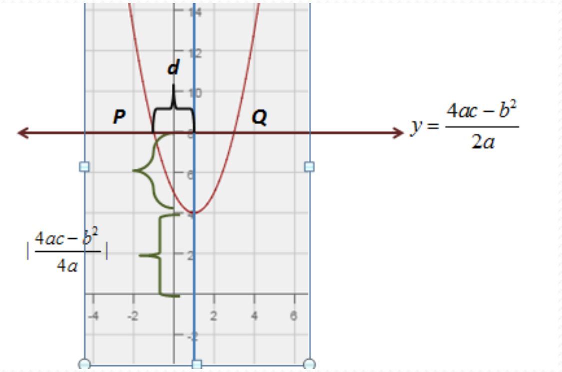
The distance from each intersection points to the axis of symmetry is 2.

So the two conjugate complex solution are the locations of the axis of symmetry which is x = 1 plus or minus the distance (d = 2) many imaginary units.

That is $x = 1 \pm 2i$

In General when D<0 and a>0

- We claim that the two complex solutions are $-\frac{b}{2a} \pm di$, which is the location of the axis of symmetry $-\frac{b}{2a}$ plus or minus ***d*** many imaginary units. (We can think that the imaginary ***i*** comes up just like that in the complex number plane, any point outside the x-axis has imaginary part.)



Algebraic Proof of Claim

Since points P and Q are the intersection points of the Horizontal line and the parabola, their x coordinates satisfy the equation:

$$\frac{4ac-b^2}{2a}=ax^2+bx+c$$

That is,

$$ax^2+bx+\frac{b^2-2ac}{2a}=0$$

Solving the equation, one gets

$$x=\frac{-b\pm\sqrt{4ac-b^2}}{2a}$$

Algebraic Proof of Claim

Hence the distance from the point ***P*** to the axis of symmetry is

$$d = \frac{\sqrt{4ac - b^2}}{2a} = \frac{\sqrt{|D|}}{2a}$$

By quadratic formulas, when D = b² – 4ac < 0, the solutions of the quadratic equation ax² + bx + c = 0, (a ≠ 0) are

$$x_1 = -\frac{b}{2a} + \frac{\sqrt{|b^2 - 4ac|}}{2a} i = -\frac{b}{2a} + di$$

and

$$x_2 = -\frac{b}{2a} - \frac{\sqrt{|b^2 - 4ac|}}{2a} i = -\frac{b}{2a} - di$$

That completes our proof of the claim.

$|\frac{4ac-b^2}{4a}|$

Remark

For the case D<o and a<o similar procedures can be used.

Instead of going up from the vertex $|\frac{4ac-b^2}{4a}|$ units along the axis of symmetry, we just go down $|\frac{4ac-b^2}{4a}|$ units, the rest of the steps are the same.

Conclusion

This procedure provides sort of visualization of the two conjugate complex roots for quadratic equations.

THANK YOU!