Rational Tangles Notes

When leading Rational Tangles for middle school students, I usually use the following outline.

- Invite four volunteers to stand in a square and hold two ropes (each about 10 ft. long) so that they form parallel edges of the square.
- This is a square dance, and there will be two dance moves that we will use: I call them "Do-Si-Do" and "Turn 'Em Round". I use the term "Do-Si-Do" to refer to the move where a student in one position lifts his or her hand and then switches places with the person standing next to them. The two positions that are swapping are always the same, though the people in those places may be different. The position of the person lifting his or her hand must always be the same. Tom Davis and some others call this move "Twist" instead or "Do-Si-Do". See the Tom Davis' video, linked below, for a vsual of what the move looks like.
- I usually use the term "Turn 'Em Round" to refer to the move where the square of people makes a quarter turn (the direction does not matter as long as it is always the same). Tom Davis usually calls this move "Rotate". He shows this move in the third video of the series linked below.
- After showing the two moves and having the students practice them a few times, I tell the students that every configuration of the ropes can be described using a rational number (a positive or negative fraction). I return us to the starting configuration with untangles ropes and ask the students to guess what number we will use to describe how tangled the ropes are. Someone usually guesses 0, which is the correct answer. I ask for a volunteer student to write that on the board.
- I tell the students that Do-Si-Do adds one to the number, while Turn 'Em Round changes the number into the negative reciprocal. We practice this using the following sequence. Every time we make a move physically, we must be careful to write down what we did on the board. We can't let the math on the board get out of step with the dancing. I usually have them write a D or a T on top of the arrow as shown below. I tell them to work with improper fractions rather than converting to mixed numbers, because it makes the negative reciprocals easier. $0 \xrightarrow{D} 1 \xrightarrow{D} 2 \xrightarrow{D} 3 \xrightarrow{T}$ -1/3 $\xrightarrow{D}$ 2/3 $\xrightarrow{D}$ 5/3 $\xrightarrow{D}$ 8/3.
- At this point, we have a bit of a tangle. I ask the dancers to display the tangle to the rest of the class. Now comes the challenge. I place a plastic grocery bag around the tangle and explain to the students that they will not be able to tell how tangled the ropes are by looking at them. Even if some of the tangle comes out of the bag, the bag will be in the way and will prevent us from knowing how untangled it really is. They must "trust the math" and make moves that get the 8/3 to turn back into 0. They may not un-Do-Si-Do or un-Turn 'Em Round. They may only do them the way shown.
- I ask the class to think ahead at each step. If we Do-Si-Do, what will happen? Starting from 8/3, adding one would make the fraction 11/3. If we Turn 'Em Round what will happen? If we take the negative reciprocal, we get -3/8. Which one will enable us to get the number back toward 0? The second option is better, though students may or may not see this. I get a student up front to be the caller – that person listens to suggestions from the class, but makes the final decision about what to do. It usually takes time for the students to learn how to untangle the ropes as efficiently as possible. Eventually, they should be able to articulate when it is better to Do-Si-Do and when it is better to Turn 'Em Round. I usually swap out the caller, dancers, and scribe at the board whenever I feel that the audience or participants might be getting tired.
- If the students do make it back to 0, then make a big deal about how we might have gotten out of step or made some mistake with our dancing or recording, or someone might have dropped their end

of the rope at some point, so who knows if this will really be untangled. Untie the bag. At first, it sometimes looks as though the ropes are tangled even though they are not. You may need to shake the ropes gently. If they did it correctly, it will be untied.

- Sometimes I try a harder example (one that gets to double digits in the numerator and/or denominator) as a group. Other times, I give pairs of ropes to teams of 6 to 8 students, have them all do the same opening sequence to get to some fraction, and then have them put bags on the tangles and try to get back to zero, recording what they did along the way. I have had some groups of students who wished to invent their own moves and then try to figure out what those moves do numerically. They usually put different test fractions on the ropes, do their move, untangle the ropes visually while recording all of the moves, and then work backwards to figure out what fraction they got right after doing the move. The question that then comes up is how can we put any given target fraction on the ropes using Ds and Ts? What is the best way to get to 5/7, for example?

Here is a video of Tom Davis demonstrating the Rational Tangles activity. This is the first of a series of 21 videos showing how the whole thing works. He moves VERY slowly in these videos – much slower than I would with students. However, he is trying to explore every aspect of the question with the teachers in his group.

https://www.youtube.com/watch?v=fRe1QEp5QvI

I like to tell the students at the end that this is a good example of a mathematical model. If they had a fairly complicated tangle in the ropes and they had to call someone on the phone and explain exactly how their friends could tangle their ropes on the same way, it would be very difficult to describe. However, if their friends knew how the Rational tangles dance works, they could just tell them to tangle until they get 8/5 on the ropes (or whatever the appropriate number is).

The fact that the number encodes a sequence of twists accurately enough to use it for untangling purposes makes it a good model for the tangled ropes. One place where tangles and knots come up in the real world when biologists are trying to understand DNA. DNA is long and stringy, and it gets tangled inside the nucleus just as uncoiled yarn stuffed in a drawer gets tangled. Many people have had the experience of trying to straighten string or rope that has been tangled in this way. DNA that is tangled can not do its job because it must untangle and unzip enough to allow the messenger RNA to be assembled so that the ribosomes can build any needed proteins using those instructions. There is an enzyme called topoisomerase which can cut a strand of DNA, move it around another strand, and then reconnect it on the other side. However, no one really knows how the topoisomerase "knows" where to cut and how to move the strands so that it untangles the DNA rather than making it worse. Rational tangles are just one among many ideas in an area of mathematics called knot theory which can be used to describe tangled and knotted strands well enough to make progress on such questions in science and engineering.