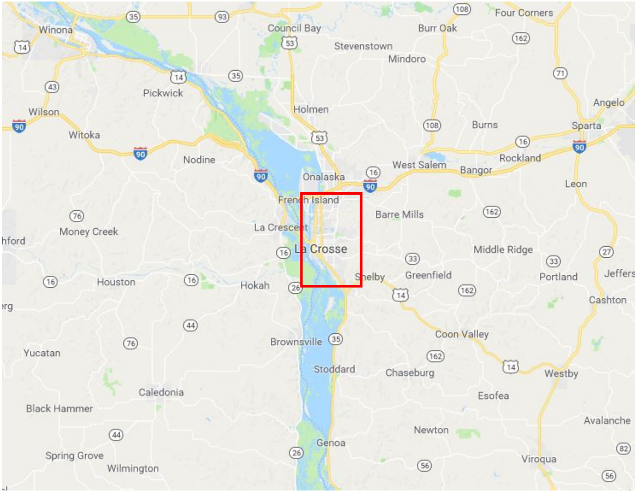
Realizing Seifert Surfaces and More!

Whitney George Dregne

https://sites.google.com/a/uwlax.edu/wgeorge/Resources

January 11, 2018

Coulee Region



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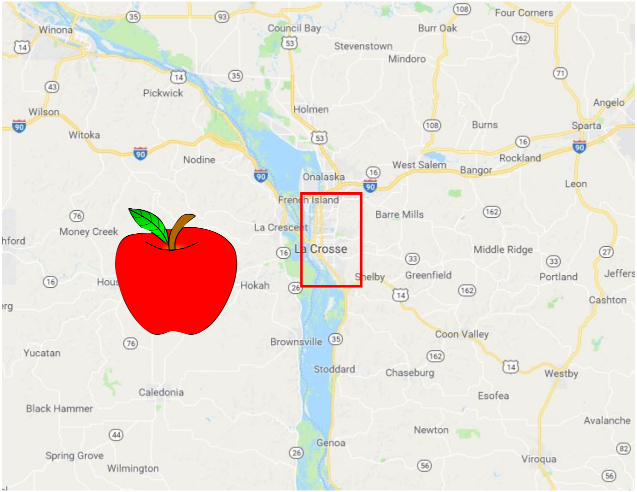

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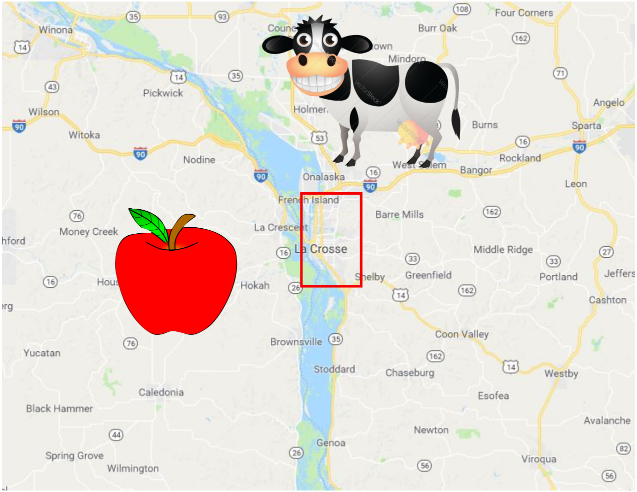

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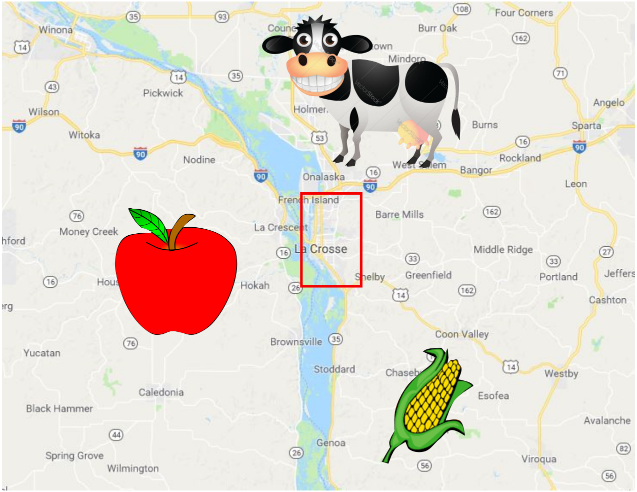

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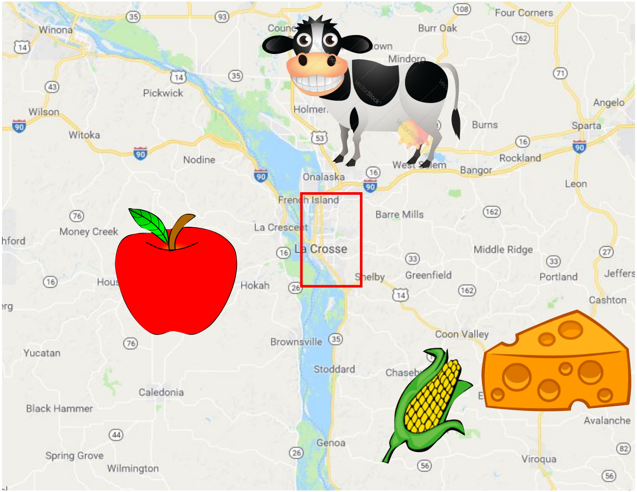

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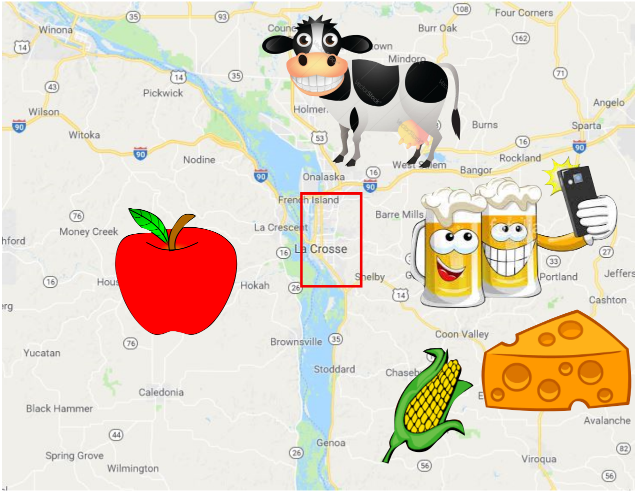
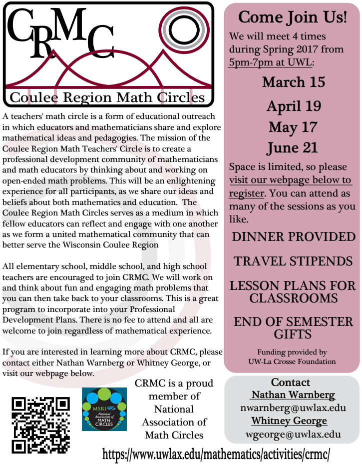


- Spring 2017 Inaugural semester



Coulee Region Math Circles

- Spring 2017 Inaugural semester
- Met every third Wednesday of each month

CRMC

Coulee Region Math Circles

A teachers' math circle is a form of educational outreach in which educators and mathematicians share and explore mathematical ideas and pedagogies. The mission of the Coulee Region Math Teachers' Circle is to create a professional development community of mathematicians and math educators by thinking about and working on open-ended math problems. This will be an enlightening experience for all participants, as we share our ideas and beliefs about both mathematics and education. The Coulee Region Math Circles serves as a medium in which fellow educators can reflect and engage with one another as we form a united mathematical community that can better serve the Wisconsin Coulee Region

All elementary school, middle school, and high school teachers are encouraged to join CRMC. We will work on and think about fun and engaging math problems that you can then take back to your classrooms. This is a great program to incorporate into your Professional Development Plans. There is no fee to attend and all are welcome to join regardless of mathematical experience.

If you are interested in learning more about CRMC, please contact either Nathan Warnberg or Whitney George, or visit our webpage below.

CRMC is a proud member of National Association of Math Circles

Come Join Us!

We will meet 4 times during Spring 2017 from 5pm-7pm at UWL:

March 15

April 19

May 17

June 21

Space is limited, so please visit our webpage below to register. You can attend as many of the sessions as you like.

DINNER PROVIDED

TRAVEL STIPENDS

LESSON PLANS FOR CLASSROOMS

END OF SEMESTER GIFTS

Funding provided by UW-La Crosse Foundation

Contact
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nwarnberg@uwlax.edu
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https://www.uwlax.edu/mathematics/activities/crmc/

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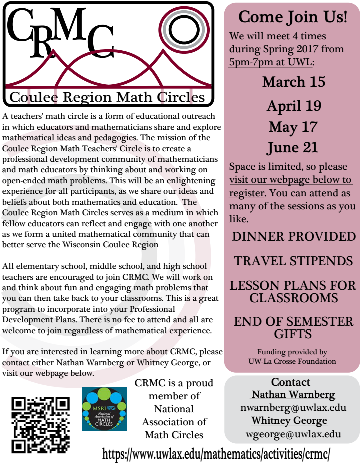

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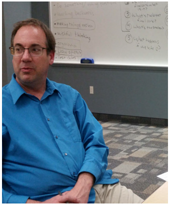
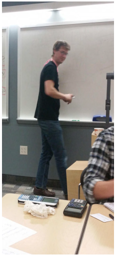
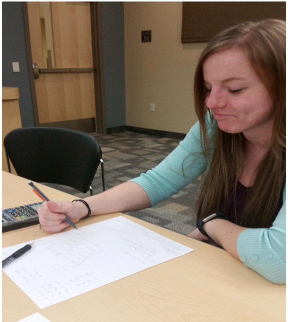
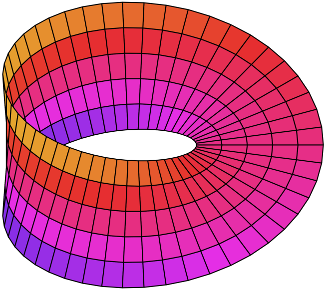


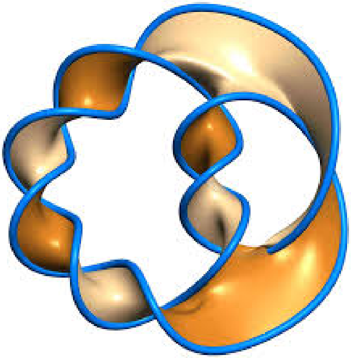




- Möbius bands



- Möbius bands
- Seifert Surfaces



- Möbius bands
- Seifert Surfaces
- Bagel cutting

Möbius bands

- Take 4 strips of paper.

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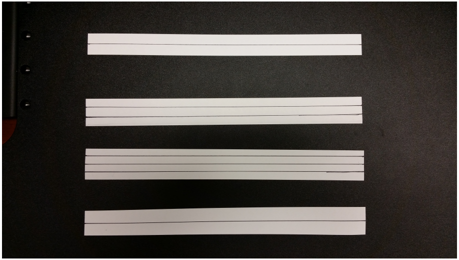
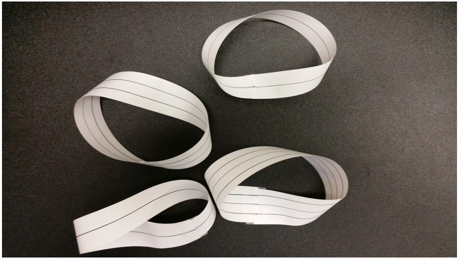


- Take 1 strip that's divided in half and twist it one full time. Then, tape the ends together

Möbius bands

- Take 1 strip that's divided in half and twist it one full time. Then, tape the ends together
- Take the other strips and twist one half time and tape the ends together



- Cut the Möbius bands along these lines.

Möbius bands

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- What happens if you divide a strip into fourths?
- What happens if you add more twists?

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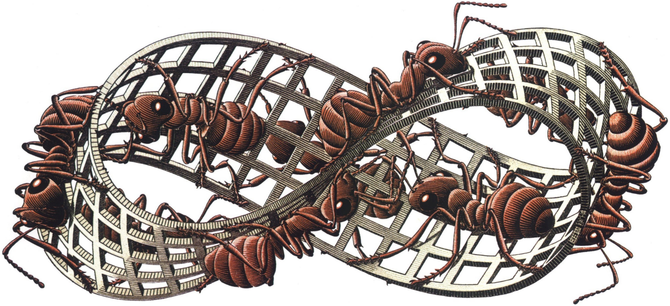

Quick Introduction to Surfaces

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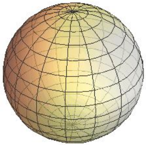
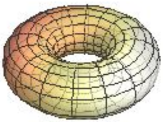
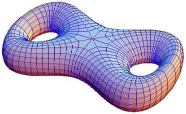
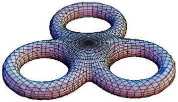

- and non-orientable surfaces

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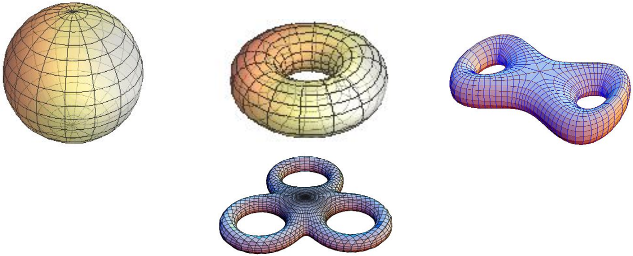

- and non-orientable surfaces
- We'll only discuss the orientable ones

Quick Introduction to Surfaces with Boundary

- imagine that you vertically slice each of these surfaces.

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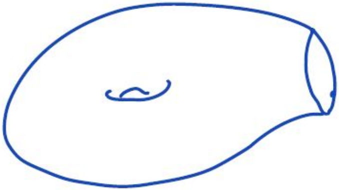

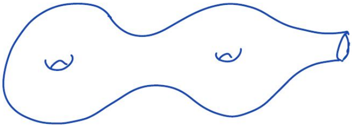

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- (Maybe not the best way to think about this...)

Introduction to Seifert Surfaces

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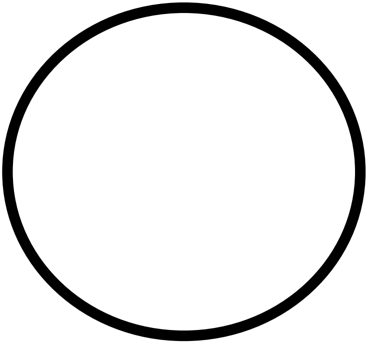

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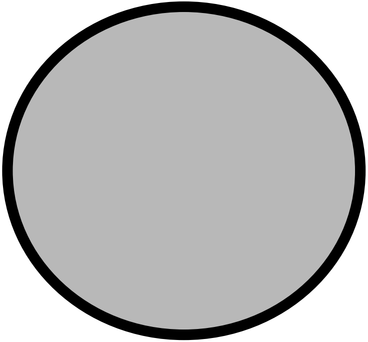

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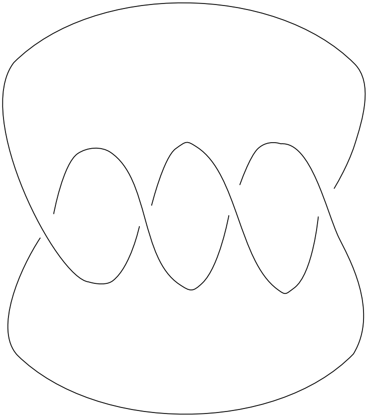

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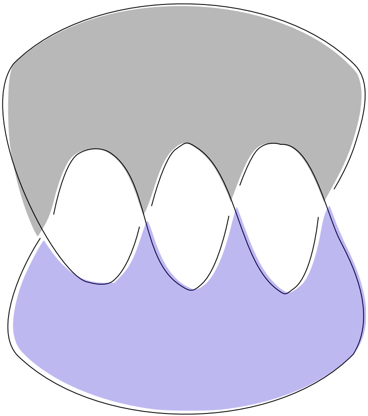

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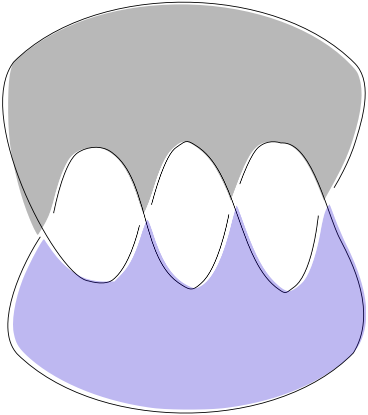
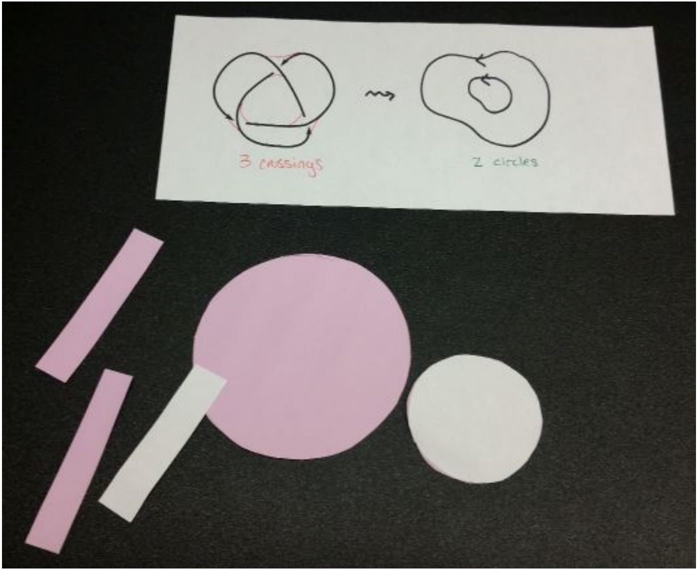


These are called Seifert Surfaces.

Creating a Seifert Surface[1]



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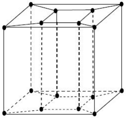

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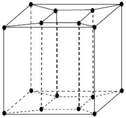

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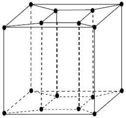

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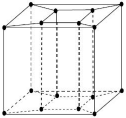

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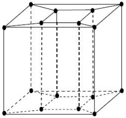

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$$\chi = 2 - 2g - 1$$

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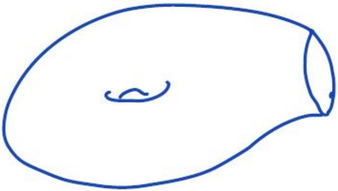

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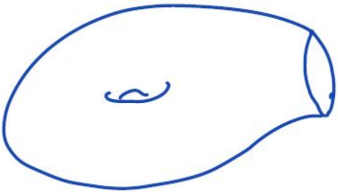

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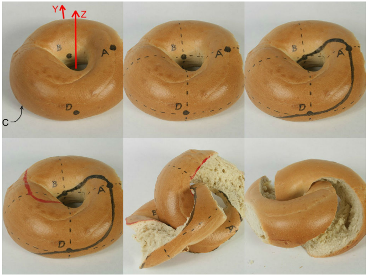
- Can knots be boundaries for non-orientable surfaces?
- How can you make sense of the fact that the trefoil can be the boundary of a torus with 1 boundary component, or edge?
- When trying to think about the trefoil being the boundary for a disk, the disk would have to intersect itself. How is this different for the Seifert surfaces? Do the Seifert surfaces intersect themselves?

- Take a bagel and draw two curves $((p, q) = (1, 1))$



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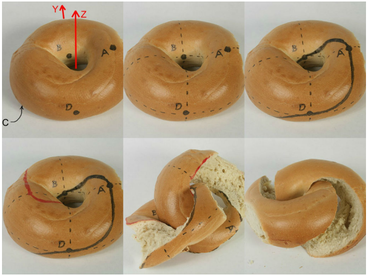

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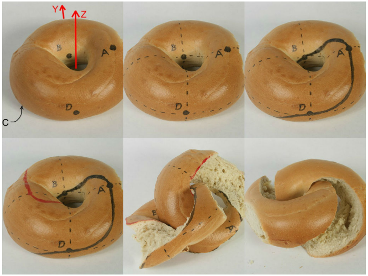
- Get a linked bagel



Bagel Cutting is fun!

Bagel Cutting-Warning

- Bagels shouldn't be pre-sliced

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- Bagels should have a hole through the middle that is relatively big
- Patience is needed....

References

https://sites.google.com/a/uwlax.edu/wgeorge/Resources

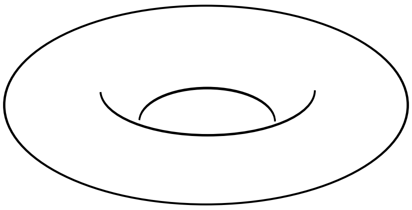
- Armstrong, A.M., Basic Topology Springer, New York, 1983
- Rolfsen, D. Knots and Links, AMS Chelsea Publishing, Rhode Island, 2003

Thanks!

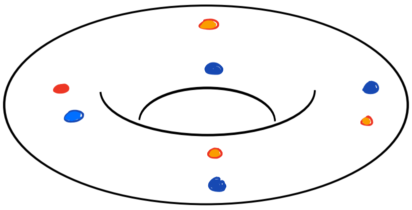
Linked Bagel

Coulee Region Math Circles | January 18, 2018

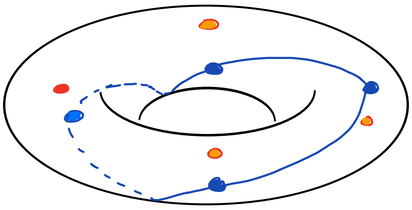
Take a bagel that has not been sliced and has a distinguishable hole



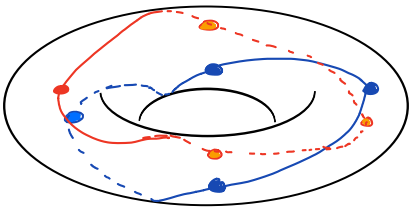
Now, draw 4 blue dots and 4 red dots as show below:



Next, use the black circles to help you draw a blue curve as shown below:



Now, draw a red curve as shown below. Make sure that the red and blue curves do NOT touch.



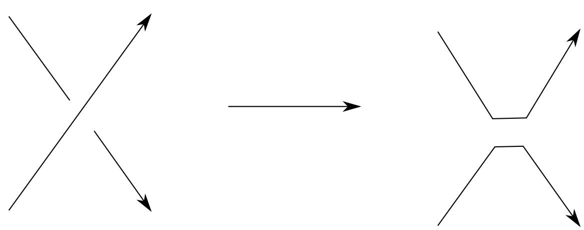
Lastly and *very carefully*, take a knife and cut along the bagel so that the knife enters the bagel along the blue curve and exits the bagel through the red curve. Once you've cut all the way around the bagel, carefully start to pull the two pieces apart.

Seifert Surfaces*

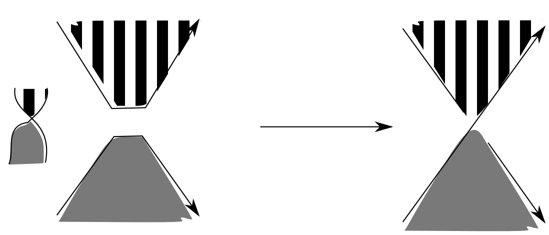
Whitney George

January 18, 2018

1. Start with the trefoil knot.
2. Near each crossing point, delete the over and under crossings and replace them with arcs so that the arrows of the knot flow together. See below:



3. You should have a disjoint collection of simple closed curves that bounds disks. For each disk, if the direction of the boundary is clockwise, assign a "-" sign. If the direction of the boundary is counter clockwise, assign a "+" sign.
4. Connect the disks together with half-twisted strips where the crossing were so that the orientation assignments agree.



Here, think about the strips as "-" and the shaded region as "+".

*This follows Rolfsen's description in Knots and Links

5. Repeat with the other 2 links/knots.

6. Compute the genus of the surface using the formula:

$$genus = 1 - \frac{s + n - c}{2}$$

   where s is the number of Seifert circles, n is the number of link components, and c is the number of crossings.