#1 IRRATIONAL PROOF

A student gives you the following proof that $\sqrt{5}$ is irrational. Give a careful, full evaluation of this argument:
We use proof by contradiction. Assume that $\sqrt{5} = \frac{n}{k}$ *where* $\frac{n}{k}$ *is a fraction in lowest terms:*

$$\sqrt{5} = \frac{n}{k} \overset{()^2=()^2}{\Longrightarrow} 5 = \frac{n^2}{k^2} \Longrightarrow 5k^2 = n^2$$

which implies that k^2 *divides* n^2*. Thus* $n \cdot n = n^2$ *is divisible by* k *since the quotient is the integer* $5k$*. However,* k *and* n *have no factors in common so* k *and* n^2 *cannot have any factors in common either.*

There are at least three criticisms of the argument which we list in increasing importance since the first two are a matter of taste while the last is essential:

- The writer of the proof should take some time to explain what it means to be rational and that n and k are both integers.
- The notion that the factors of n are the same as the factors of n^2 needs some explanation. This makes sense when we explain that we mean prime factors and that each integer has (except for ordering) a unique prime factorization. Thus if $n = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$ (i.e. the full set of prime factors of n is $\{p_1, p_2, \ldots, p_k\}$) then since $n^2 = p_1^{2s_1} p_2^{2s_2} \cdots p_k^{2s_k}$, the set of prime factors of n^2 is the same.)
- We need to separately rule out the possibility that $\sqrt{5}$ is an integer. We can do this for example by noting that $2^2 < 5 < 3^2$. This is essential because if this were not so then we could have $k = 1$. Then k does divide n and hence k^2 also divides n^2 without contradiction.

#2 THE TAX MAN

The tax man is a game played solo against the fictitious tax man. You start with the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. You take turns picking valid numbers from this list and removing them. On each turn the tax man gets all divisors of the number you chose that still remain. You cannot choose any number that doesn't have at least one divisor still remaining. When no moves are possible, all remaining numbers go to the tax man. You win provided the sum of the numbers you have exceeds the sum of those held by the tax man.

Either find a way to beat the tax man or explain why this is impossible.

Here is a rout of the tax man by a score of 50 to 28. There may be other winning plays.

REMAINING	**YOURS**	**TAX MAN's**	**CHOICE**
$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$	$\{\}$	$\{\}$	11
$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 12\}$	$\{11\}$	$\{1\}$	9
$\{2, 4, 5, 6, 7, 8, 10, 12\}$	$\{9,11\}$	$\{1,3\}$	8
$\{5, 6, 7, 10, 12\}$	$\{8,9,11\}$	$\{1,2,3,4\}$	10
$\{6, 7, 12\}$	$\{8,9,10,11\}$	$\{1,2,3,4,5\}$	12
$\{7\}$	$\{8,9,10,11,12\}$	$\{1,2,3,4,5,6\}$	**NO MOVES**
$\{\}$	$\{8, 9, 10, 11, 12\} \Longrightarrow 50$	$\{1, 2, 3, 4, 5, 6, 7\} \Longrightarrow 28$	

#3 SHUT THE BOX or SHUT UP

The game shut up is played solo with a pair of ordinary dice and the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. On each roll of the dice you may eliminate any set of numbers whose sum equals the number rolled. You continue rolling until you find a roll for which this cannot be done. Your score is then the sum of the numbers still remaining. **You want the lowest score possible - shutting the box is a score of zero!**.

Suppose you are playing shut the box and the numbers remaining are just $1, 2, 6,$ *and* 7 *and you roll a* 5 *&* 3. *What should you do and why?*

[1] The options for the 8 are 7 & 1 or 6 & 2. Both choices reduce the total of the remaining numbers to 8. Furthermore, both options lead to a probability that another move can be made beyond this of $\frac{11}{36}$:
In the 7 & 1 case the next roll needs to be any one of 2 $\left(\frac{1}{36}\right)$ or 6 $\left(\frac{5}{36}\right)$ or 8 $\left(\frac{5}{36}\right)$ and $\frac{1}{36}+\frac{5}{36}+\frac{5}{36} = \frac{11}{36}$.
In the 6 & 2 case the next roll needs to be any one of 7 $\left(\frac{6}{36}\right)$ or 8 $\left(\frac{5}{36}\right)$ and $\frac{6}{36} + \frac{5}{36} = \frac{11}{36}$.

We look closer at these options by asking about the chances of closing the box after this move:

- Choosing 7 & 1 and leaving $2, 6$ means the box can be closed by one of the following routes:
  - Roll 2 $\left(\frac{1}{36}\right)$ and then roll a 6 $\left(\frac{5}{36}\right)$ which has probability $\left(\frac{1}{36}\right)\left(\frac{5}{36}\right) = \frac{5}{1296}$.
  - Roll 6 $\left(\frac{5}{36}\right)$ and then roll a 2 $\left(\frac{1}{36}\right)$ which has probability $\left(\frac{5}{36}\right)\left(\frac{1}{36}\right) = \frac{5}{1296}$.
  - Roll an 8 with probability $\frac{5}{36}$

  So the total chance of closing the box are $\frac{5}{1296} + \frac{5}{1296} + \frac{5}{36} = \frac{95}{648}$.
- Choosing 6 & 2 and leaving $1, 7$ means the box can be closed in only one way:
  - Roll 7 with probability $\left(\frac{6}{36}\right)$ but then there is no way to roll a one and your score is 1.
  - Roll an 8 with probability $\frac{5}{36}$

  So the total chance of closing the box is just $\frac{5}{36}$ which is less than the above.

However, it is clearly better to choose 6 & 2 because then the chances of a score of at most one becomes $\frac{6}{36} + \frac{5}{36} = \frac{11}{36}$ whereas in the case of choosing 7 & 1 the probability of at most 1 is the same as the probability of closing the box and this we found as $\frac{95}{648}$.

[1] In some versions of the game when the points remaining is small enough the rules allow one the option of rolling just one die. That option would make the chances of closing the box the same with either choice.

#4 ARRANGING DIGITS

Use each digit $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ exactly once for the letters $a, b, c, d, e, f, g, h, i$ so that the following has the smallest possible absolute value. How do you know your answer is the smallest and what is this smallest value?

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} - i$$

It's actually possible to get zero! Such a result requires the numbers to combine well. By this we mean that when reduced, the fractional (i.e. non-integer) numbers formed should have simple denominators and a simple common denominator like 2 or 4 (or maybe 3). In the first solution below therefore, we pair 5 and 4, 3 and 6, 2 and 8, 7 and 1, and leave 9 unpaired. As we see, other pairings with either 9 or 7 unpaired are also possible:[2]

$$\frac{5}{4} + \frac{3}{6} + \frac{7}{1} + \frac{2}{8} - 9 = 1.25 + 0.5 + 7 + 0.25 - 9 = 2 + 7 - 9 = 9 - 9 = 0$$
$$\frac{7}{4} + \frac{5}{1} + \frac{6}{3} + \frac{2}{8} - 9 = 1.75 + 5 + 2 + 0.25 - 9 = 2 + 5 + 2 - 9 = 0$$
$$\frac{6}{8} + \frac{2}{1} + \frac{5}{4} + \frac{9}{3} - 7 = 0.75 + 2 + 1.25 + 3 - 7 = 2 + 2 + 3 - 7 = 0$$

Other pairings using that $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ and leaving out $5, 8$, or 9 (again) are also possible:

$$\frac{7}{6} + \frac{5}{2} + \frac{4}{1} + \frac{3}{9} - 8 = 1\frac{1}{6} + \frac{1}{3} + 2\frac{1}{2} + 4 - 8 = 1\frac{1}{2} + 2\frac{1}{2} + 4 - 8 = 4 + 4 - 8 = 0$$
$$\frac{6}{9} + \frac{7}{2} + \frac{4}{8} + \frac{1}{3} - 5 = \frac{2}{3} + \frac{1}{3} + 3\frac{1}{2} + \frac{1}{2} - 5 = 1 + 4 - 5 = 0$$
$$\frac{2}{8} + \frac{1}{6} + \frac{9}{4} + \frac{7}{3} - 5 = \frac{1}{4} + \frac{1}{6} + 2\frac{1}{3} + 2\frac{1}{4} - 5 = 2.5 + 2.5 - 5 = 0$$
$$\frac{5}{6} + \frac{4}{8} + \frac{2}{3} + \frac{7}{1} - 9 = \left(\frac{5}{6} + \frac{2}{3}\right) + \frac{4}{8} + 7 = 1.5 + 0.5 + 7 - 9 = 2 + 7 - 9 = 0$$

#5 ALPHABETIZING

Consider all the integers from 1 to 10^{10} and suppose we write each of these in English. For example, Three Billion, Four Million, Fifty Eight Thousand and Eighteen (i.e. 3,004,058,018). Ignoring all spaces, commas, hyphens (if included), and conjunctions, alphabetize this list. Thus 3,004,058,018 would be simply ThreeBillionFourMillionFiftyEightThousandEighteen where the capitals are given for emphasis only. What is the first odd number in the list and where does this occur?

[2] Except for the ordering of the factors, this listing is complete.

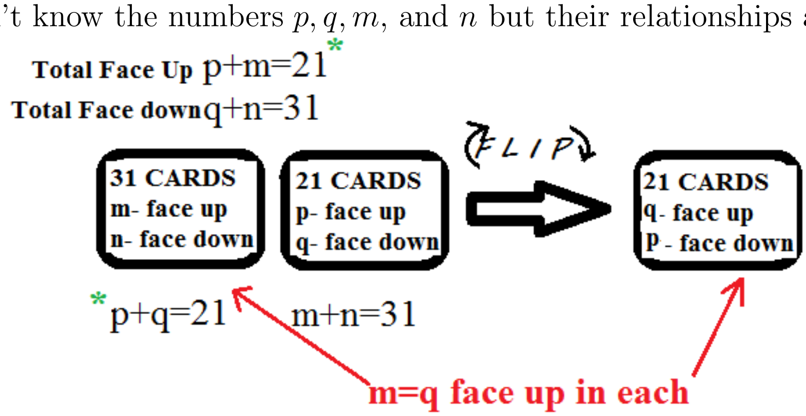
Alphabetically, the first 14 numbers are as follows:
(1) 8 Eight
(2) 8,000,000,000 EightBillion
(3) 8,000,000,008 EightBillionEight
(4) 8,000,000,018 EightBillionEighteen (Notice EighteenBillion 18,000,000,000 is too big a number)
(6) 8,018,000,000 EightBillionEighteenMillion
(7) 8,018,000,008 EightBillionEighteenMillionEight
(8) 8,018,000,018 EightBillionEighteenMillionEighteen
(9) 8,018,018,000 EightBillionEighteenMillionEighteenThousand
(10) 8,018,018,008 EightBillionEighteenMillionEighteenThousandEight
(11) 8,018,018,018 EightBillionEighteenMillionEighteenThousandEighteen
(12) 8,018,018,080 EightBillionEighteenMillionEighteenThousandEighty
(13) 8,018,018,088 EightBillionEighteenMillionEighteenThousandEightyEight
(14) 8,018,018,085 EightBillionEighteenMillionEighteenThousandEightyFive

Thus the first odd is 8,018,018,085 in 14th position.

#6 CARD TRICK

You are placed in a completely dark room with no flashlight. In other words, you cannot see anything! On a table spread out in front of you are 52 playing cards. 21 of these are face up and 31 are face down. Your job is to create two piles of cards which both have the same number of face up cards. How can you do it?

The problem is mystifying at first since the total number of face up cards is odd and hence, as it stands, cannot be divided evenly. Thus we have to change those numbers and the following procedure works: Since we can feel and count in the dark we start by making two piles of cards - one of size 31 and the other of size 21. This is not the face up/face down sorting but it helps the math! In fact, all you have to do is turn over the 21 card pile and you are done! In the following explanation we don't know the numbers p, q, m, and n but their relationships are as shown:



From the equations with an asterisk $*$ we have

$$p + m = 21 = p + q \overset{-p}{\Longrightarrow} m = q = \#\text{ Face up in each pile after flipping}$$

You can check of course that there is nothing special about the numbers 21 and 31 except that they add to 52. In addition, we could flip either pile and both piles also have the same number of face down cards.