Bounds on Solvable Snake Cube Puzzle

Anthony Bosman
with Adrian Negrea

Andrews University
Seek. Affirm. Change.

August 5, 2021

Snake Cube Puzzle

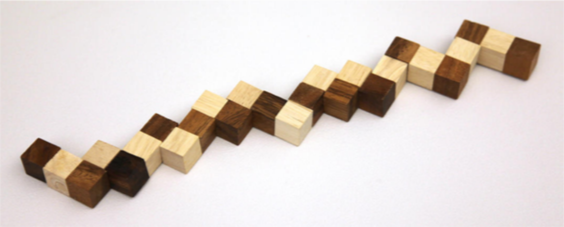
The snake cube puzzle is solved by folding a snake-like chain of connected pieces into an $n \times n \times n$ cube.

A $3 \times 3 \times 3$ snake cube.

A great source of interesting mathematical questions! (eg. Undergraduate Research)

Standard Position

A snake puzzle can be unfolded into a standard position, moving only rightward and upward. This gives a way of encoding any puzzle.

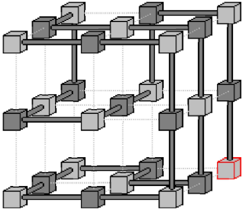


The pattern **ESTTTSTTSTTTSTSTTTTSTSTSTSE** where

- **E:** denotes one of the two end pieces,
- **S:** denotes a straight piece, that is, a piece where the elastic cord exits through the face opposite to the face where it entered,
- **T:** denotes a twisted piece, that is, a piece where the elastic cord exits through a face adjacent to the face where it entered.

When can a snake be folded into a cube?

When can a snake be folded into a cube?



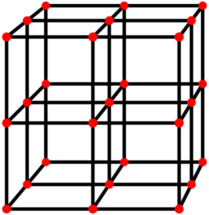
No Solution

What are some strategies to determine when there is a solution?

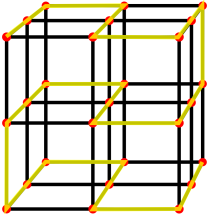
Grid Graphs & Hamiltonian Paths

The snake cube has been studied as a Hamiltonian path in a grid graph [Itai 1982; Zamfirescu 1992].

A $3 \times 3 \times 3$ grid graph:



A Hamiltonian path hits every vertex in the graph exactly once:



Theorem [Abel, Demaine, et. al. 2012]

It is NP-complete to decide when a snake puzzle can be folded into an $n \times n \times n$ cube.

Theorem [Abel, Demaine, et. al. 2012]

It is NP-complete to decide when a snake puzzle can be folded into an $n \times n \times n$ cube.

Idea of Proof

Reduce the 3-partition problem (known to be NP-complete) to determining if there is a solution to a snake cube puzzle.

When can a set of $3n$ numbers be partitioned into n triplets that all have the same sum?

$$\{20,23,25,49,45,27,40,22,19\}$$
$$\downarrow$$
$$\begin{array}{ccc} \{20,25,45\}, & \{23,27,40\}, & \{49,22,19\} \\ 20+25+45=90, & 23+27+40=90, & 49+22+19=90 \end{array}$$

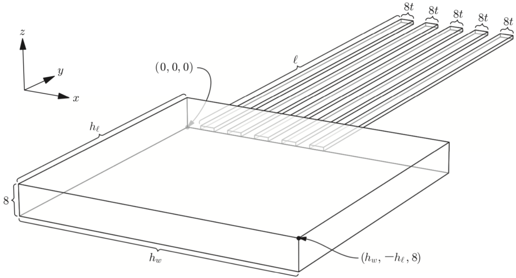
Reduction to 3-Partition Problem

Suppose we are given a set of integers $S = \{x_1, x_2, ..., x_{3n}\}$. From S we build a snake cube puzzle $P(S)$ such that S has a 3-partition exactly when $P(S)$ folds into a cube.

Reduction to 3-Partition Problem

Suppose we are given a set of integers $S = \{x_1, x_2, ..., x_{3n}\}$. From S we build a snake cube puzzle $P(S)$ such that S has a 3-partition exactly when $P(S)$ folds into a cube.

Building $P(S)$: first part leaves "hub-and-slots", second comprises subpuzzles that can only fill slots if partitions, last fills the remainder of the hub.



Our Approach

An invariant is a feature of the puzzle that does not change as it is folded.

- The number of components of the puzzle: for it to fold into an $n \times n \times n$ cube, must begin with n^3 piece puzzle.
- The maximal number of straights in a row: a puzzle cannot fold into an $n \times n \times n$ cube if it has more than $n - 2$ straight pieces in a row.

Question: Can we identify another invariant(s) that will help to determine when a snake cube puzzle is solvable?

Some Data

Number of Solutions	Number of Straights: 2	3	4	5	6	7	8	9	10	11	Total
1	0	15	144	589	1053	1078	556	187	34	2	**3658**
2	0	26	145	502	862	770	325	65	9	0	**2704**
3	0	25	118	326	393	255	104	21	0	0	**1242**
4	1	14	98	242	340	203	89	14	1	0	**1002**
5	0	12	56	140	168	86	12	1	0	0	**475**
6	1	11	57	123	181	97	23	3	0	0	**496**
7	2	6	36	101	71	30	0	0	0	0	**246**
8	0	8	35	89	83	43	22	7	1	0	**288**
9	3	9	23	51	61	17	5	0	0	0	**169**
10	0	10	33	57	34	16	6	0	0	0	**156**
11	0	6	28	39	22	11	1	1	0	0	**108**
12	2	4	30	39	35	17	9	1	0	0	**137**
13	2	2	20	31	9	5	1	0	0	0	**70**
14	0	6	23	25	24	10	0	0	0	0	**88**
15	0	2	16	28	12	3	0	0	0	0	**61**
16	0	3	15	24	11	7	5	3	1	0	**69**
17	1	9	14	16	9	3	0	0	0	0	**52**
18	1	3	9	6	10	7	1	0	0	0	**37**
19	0	6	12	16	6	2	0	0	0	0	**42**
20	0	1	11	18	11	1	0	0	0	0	**42**

Brute force analysis from Jaap's puzzle age (https://www.jaapsch.net/puzzles/snakecube.htm)

Some Data

		Number of Straights										
		2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**Total**
	1	0	15	144	589	1053	1078	556	187	34	2	**3658**
N	**2**	0	26	145	502	862	770	325	65	9	0	**2704**
u	**3**	0	25	118	326	393	255	104	21	0	0	**1242**
m	**4**	1	14	98	242	340	203	89	14	1	0	**1002**
b	**5**	0	12	56	140	168	86	12	1	0	0	**475**
e	**6**	1	11	57	123	181	97	23	3	0	0	**496**
r	**7**	2	6	36	101	71	30	0	0	0	0	**246**
	8	0	8	35	89	83	43	22	7	1	0	**288**
o	**9**	3	9	23	51	61	17	5	0	0	0	**169**
f	**10**	0	10	33	57	34	16	6	0	0	0	**156**
	11	0	6	28	39	22	11	1	1	0	0	**108**
S	**12**	2	4	30	39	35	17	9	1	0	0	**137**
o	**13**	2	2	20	31	9	5	1	0	0	0	**70**
l	**14**	0	6	23	25	24	10	0	0	0	0	**88**
u	**15**	0	2	16	28	12	3	0	0	0	0	**61**
t	**16**	0	3	15	24	11	7	5	3	1	0	**69**
i	**17**	1	9	14	16	9	3	0	0	0	0	**52**
o	**18**	1	3	9	6	10	7	1	0	0	0	**37**
n	**19**	0	6	12	16	6	2	0	0	0	0	**42**
s	**20**	0	1	11	18	11	1	0	0	0	0	**42**

Brute force analysis from Jaap's puzzle age (https://www.jaapsch.net/puzzles/snakecube.htm)

Question: Can we give a proof for why the maximal number of straights is 11 in a solvable $3 \times 3 \times 3$ cube and generalize to $n \times n \times n$?

Counting Non-Straights

We observe:

- Each face must have at least $2n$ non-straight pieces (two per row/column),
- But this triple counts the 8 corner pieces since each corner belongs to three faces,
- And this double counts $n-2$ edge pieces on each edge,
- Note there are at most four non-adjacent edges in cube.

Therefore, we have that the number of non-straight pieces is at least

$$\underbrace{12n}_{\text{total}} - \underbrace{8(2)}_{\text{corners}} - \underbrace{4(n-2)}_{\text{edges}} = 8n - 8.$$

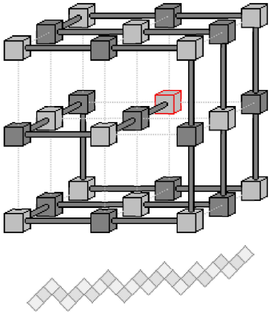
Let $S_{max}(n)$ denote the maximum number of straight components in an $n \times n \times n$ solvable snake cube puzzle. Then we have shown:

$$S_{max}(n) \leq n^3 - (8n - 8) = n^3 - 8n + 8.$$

In particular, we have:

$$\begin{array}{rcl} S_{max}(3) & \leq & 11 \\ S_{max}(4) & \leq & 40 \\ S_{max}(5) & \leq & 93 \end{array}$$

Lower Bound on Maximal Straights



Generalizing the above pattern, there exists a snake of n^3 pieces that has

$$\underbrace{n(n-2)}_{\text{top layer}} + \underbrace{(n-2)[(n-1)(n-2)+n]}_{\text{middle }(n-2)\text{ layers}} + \underbrace{n(n-2)}_{\text{bottom layer}} = n^3 - 2n^2 + 2n - 4$$

straight pieces, giving a lower bound on the maximal number of straights.

Thus, we have shown:

	$n^3 - 2n^2 + 2n - 4$	$\leq s_{\text{max}} \leq$	$n^3 - 8n + 8$	n^3
$n = 3$	11	$\leq s_{\text{max}} \leq$	11	27
$n = 4$	36	$\leq s_{\text{max}} \leq$	40	64
$n = 5$	81	$\leq s_{\text{max}} \leq$	93	125
$n = 6$	152	$\leq s_{\text{max}} \leq$	176	216

Notice the size of the gap is quadratic: $2n^2 - 10n + 12$.

Some Related Questions

- Can we precisely determine $S_{max}(n)$ for $n > 3$ rather than have a range,
- Investigate $S_{min}(n)$, the minimal number of straights in an $n \times n \times n$ snake cube (known to be zero for n even).

Other Snake Research

- Rusky and Sawada (2003): Generalized setting to puzzles filling $n_1 \times n_2 \times n_3$ box, as well to d dimensions and toridial space.

Theorem (Rusky, Sawada)

If $d \geq 3$ and at least one n_i is even then there is a bent Hamiltonian cycle that fills $n_1 \times n_2 \times \cdots \times n_d$.

- McDonough (2009, Undergrad Honors Project, Smith): Analyzed cube Hamiltonian paths that formed knots. Which knots can be formed in this way? What is the smallest cube needed to achieve a given knot (or link)?

Thank You!

Anthony Bosman
bosman@andrews.edu

Andrews University
Seek. Affirm. Change.