Calculus Favorite: Stirling's Approximation, Approximately

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Introduction

- Stirling approximation says $n! \approx \sqrt{2\pi n}\,(\frac{n}{e})^n$ as $n \to \infty$ in the asymptotic sense that the ratio has limit 1.
- Nice topic for lots of reasons: challenging but reachable; important; useful in calculus (power series) and in computer science / discrete optimization (complexity of brute force search); backwards thinking – approximating sum by integral; leave a bit to be done – extends
- In this short presentation, will try to give flavor of class – done with GMU honors and regular calculus 2; BC at magnet school as special appearance; in higher level TJ courses as an aside. Done interactively in class.

Start of class discussion

- **How big is $n!$?**
- **Say for $n = 1000$?**
- **We'll do this in computer algebra.**

Fun begins

From Mathematica: input is Factorial[1000] which outputs as

Crazy output

40238726007709377354370243392300398571937486421071463254379991042993851239862902059204420848696940480047
99886101971960586316668729948085589013238296699445909974245040870737599188236277271887325197795059509952
7612087497546249704360141827809464649629105639388743788648733711918104582578364784997701247663288983595573543251318532395846307555740911426241747434934755342864657661166779739666882029120737914385371958824980812686783837455973174613608537953452422158659320192809087829730843139284440328123155861103697680135730421616874760967587134831202547858932076716913244842623613141250878020800026168315102734182797770478463586817016436502415369139828126481021309276124489635992870511496497541990934222156683257208082133318611681155361583654698404670897560290095053761647584772842188967964624494516076535340819890138544248798495995331910172335555660213945039973628075013783761530712776192684903435262520001588853514733161170210396817592151090778801939317811419454525722386554146106289218796022383897147608850627686296714667469756291123408243920816015378088989396451826324367161676217916890977991190375403127462228998800519544441428201218736174599264295658174662830295557029902432415318161721046583203678690611726015878352075151628422554026517048330422614397428693306169089796848259012545832716822645806652676995865268227280707578139185817888965220816434834482599326604336766017699961283186078838615027946595513115655203609398818061213855860030143569452722420634463179746059468257310379008402443243846565724501440282188525247093519062092902313649327349756551395872055965422874977401141334696271542284586237738753823048386568897646192738381490014076731044664025989949022

Crazy output – Continued

22217659043399018860185665264850617997023561938970178600408118897299183110211712298459016419210688843871
21855646124960798722908519296819372388642614839657382291123125024186649353143970137428531926649875337218
94069428143411852015801412334482801505139969429015348307764456909907315243327828826986460278986432113908
35062170950025973898635542771967428222487575867657523442202075736305694988250879689281627538488633969099
59826280956121450994871701244516461260379029309120889086942028510640182154399457156805941872748998094254
74217358240106367740459574178516082923013535808184009699637252423056085590370062427124341690900415369010
59339838357779394109700277534720000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000

Laughter subsides, now floating point version

From Mathematica: input is N[Factorial[1000]] which outputs as

$4.023872600770938 \times 10^{2567}$

Try to explain this – often get something like 1000 terms, average value 500, so roughly

$$500^{1000}$$

This is $9.33263618503219 \times 10^{2698}$ (spared you the total output)

Only off by about 2×10^{131} – not too bad?!

Further exploration begins

Soon get discussion to ask about $n!$ and its definition as repeated multiplication

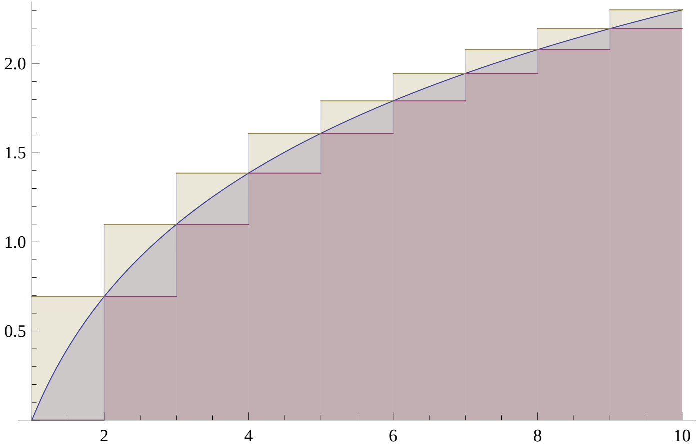
Consider the summation idea using ln(n!)

$$\ln(n!) = \sum_{k=1}^{n} \ln(k)$$

and now compare sum to integral using left and right endpoints (and soon midpoint).

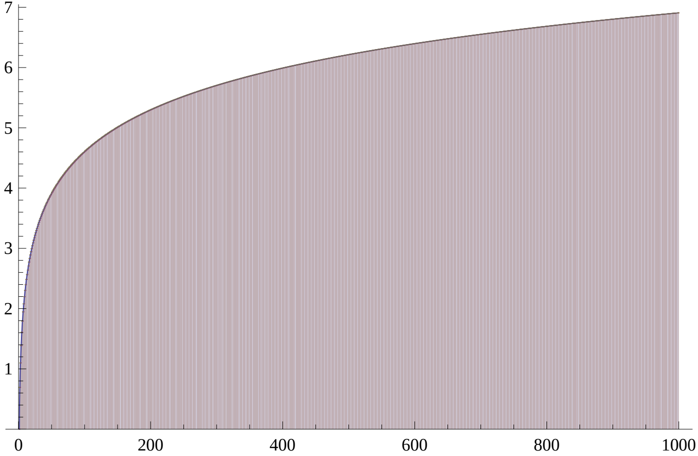
Graphical view

Here is a graph, using 10 instead of 1000 so we can see things.



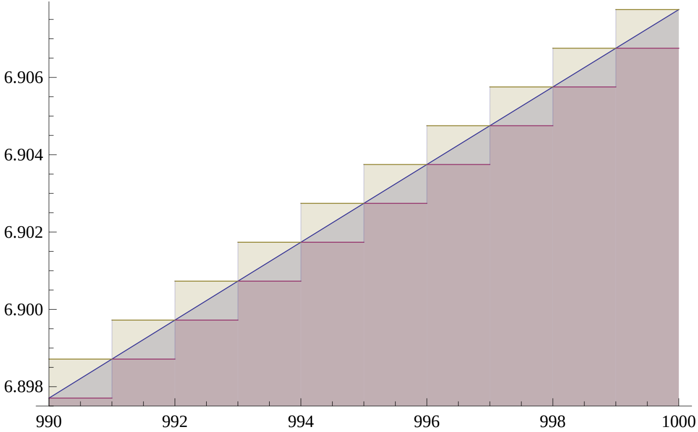
More graphics

Here is the graph, using 1000 steps where we can't see as well.



More graphics still

Here is the tail of the graph, using 1000 steps, showing last ten.



Some analysis

As a first attempt, consider the integral of $\ln(x)$, compared to the Riemann left and right sums:

$$\int_1^n \ln(x)\,dx = x\,\ln(x) - x\big|_{x=1}^{x=n} = n\,\ln(n) - n + 1$$

Graph increases, so left endpoint sum is lower, right endpoint is higher. This yields some estimates:

$$\ln(n!) - \ln(n) = \sum_{k=1}^{n-1} \ln(k) < \int_1^n \ln(x)\,dx < \sum_{k=2}^{n} \ln(k) = \ln(n!)$$

Analysis continued

The inequalities we obtain for $\ln(n!)$ are not fabulous (yet):

$$n \ln(n) - n + 1 < \ln(n!) < n \ln(n) - n + 1 + \ln(n)$$

This yields the initial estimates:

$$(n/\mathrm{e})^n \, \mathrm{e} < n! < (n/\mathrm{e})^n \, n \, \mathrm{e}$$

Trapezoid approximation

Using the trapezoid approximation rather than endpoints does a better job (average of left and right)

$$\int_1^n \ln x \, dx \approx \sum_{k=2}^{n} \left(\frac{\ln(k-1)+\ln(k)}{2}\right) = \ln(n!) - \frac{1}{2}\ln(n)$$

This unrolls to the approximation (note: arithmetic mean of logs is geometric mean without logs):

$$n! \approx (n/\mathrm{e})^n \mathrm{e}\sqrt{n}$$

Correct except numerical factor: e vs. $\sqrt{2\pi}$.

Frosting on the cake

Numerical values are as follows

$$\mathrm{e} \approx 2.718281828459045$$

$$\sqrt{2\pi} \approx 2.5066282746310002$$

Correct except numerical factor of about 10%.

Full expansion can be had with some extra effort (Euler-Maclaurin formula).

Fancy script writing on the frosting on the cake

From graphs it is clear most of the error is in the early terms. Using the discrete sum for a few steps and then using the integral cuts the numerical discrepancy. Here are some easy first few steps:

$$\begin{aligned} ((e*2)/3)^{\frac{3}{2}} &\approx 2.439522535141459 3 \\ 2*(2*e/5)^{\frac{5}{2}} &\approx 2.465563423812403 \\ 2*3*(2*e/7)^{\frac{7}{2}} &\approx 2.477101383175650 \end{aligned}$$

Recall

$$\sqrt{2\pi} \approx 2.5066282746310002$$

Correct except numerical factor of about 1%.

Using this result in teaching series

When teaching power series, I use this result a lot to help students guess (intelligently) about radius of convergence.

From Taylor (or Maclaurin) – another naming issue – we want to understand the sums:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n!}$$

and Stirling says $n!$ is much larger than the exponential term $(x-a)^n$, so it boils down to the growth rate (in n) of the derivatives.

For exponentials and basic trig (sine and cosine) the factorial wins and the radius is infinite, while for fractional and negative powers, and their integrals or derivatives, there is a balance and a finite radius.

The exact value for the leading constant term

The leading term (and more) can be obtained with some effort using Laplace's method for asymptotic expansions, which I suggest to students that they take more math.

History is also pretty interesting: Stirling got this constant but deMoivre did most of the work, but not the honor of the naming. Used Wallis' product formula.

Came in the context of probability for repeated Bernoulli trials, fair coin. $2n$ tries, exactly n of heads, tails. Need the central binomial coefficient:

$$\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}$$

which Stirling's formula will approximate well and give the important factor of $n^{-\frac{1}{2}}$. DeMoivre got the Gaussian (bell curve) out of the approximation.

The full asymptotic expansion can be done by Laplace's method, starting from the formula $n! = \int_0^\infty t^n \, e^{-t} \, dt$.

Concluding remarks

This is both a pretty and a useful result.

The mathematics is not deep, but there is considerable thought involved.

Lots of pieces came into play: integration by parts, integral related to discrete sum, crux move in problem solving.

Thank you for your attention and I welcome your comments and/or questions.