Mathematical Explorations with Swish

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North Andover, MA

MathFest
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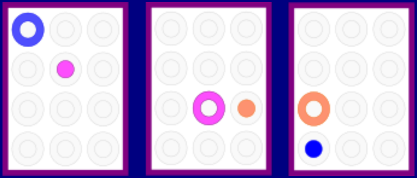
Outline

1. Overview of Swish
   - How to Play
   - Transformational geometry
2. Swish in the college classroom
   - Combinatorics and Group theory
   - Discrete Probability
   - Analogies to linear algebra
3. Open questions

What is Swish?



- Cards are transparent.
- Each card contains one hoop (annulus) and one ball (disk).

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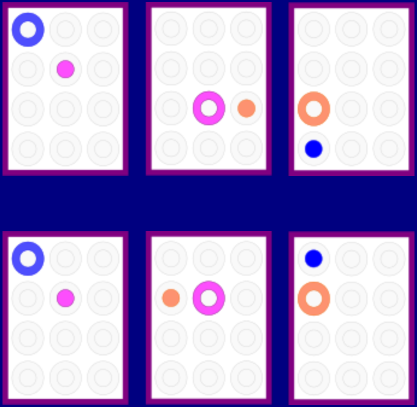

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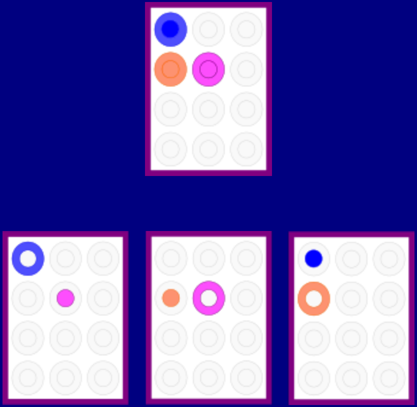
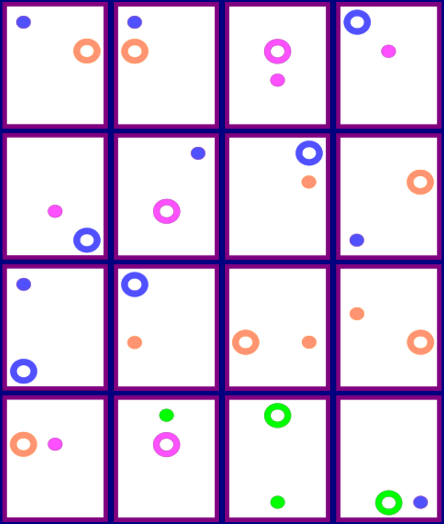


- Cards are transparent.
- Each card contains one hoop (annulus) and one ball (disk).
- Cards can be rotated or flipped over to align the balls and hoops.
- A swish occurs when a subset cards are stacked so every hoop is filled with a ball.

How many swishes can you find?



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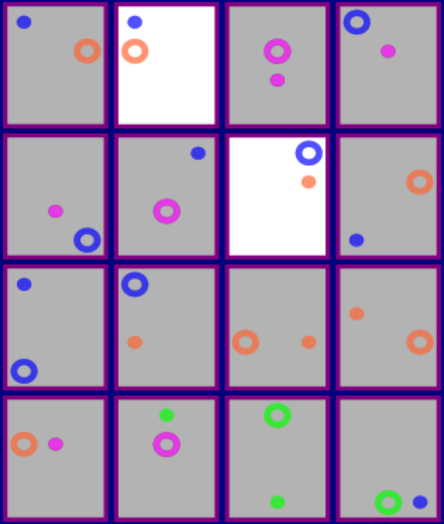

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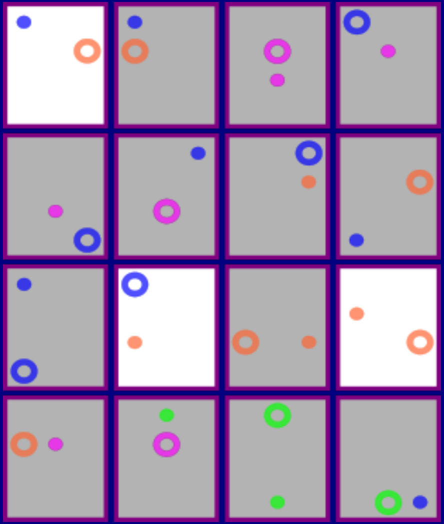

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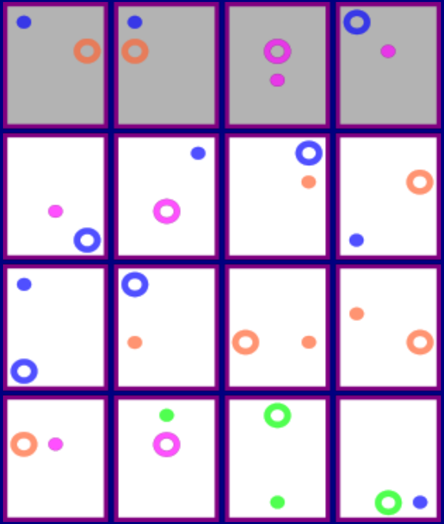
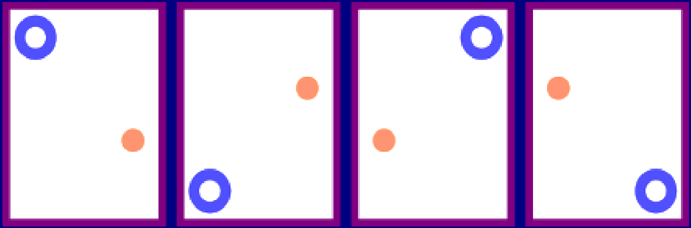


Why Swish?

- Created by Gali Shimoni and Zvi Shalem (Israel Center for Excellence through Education) as a way to explore transformational geometry
- Each card can be transformed in four ways: identity e, vertical flip v, horizontal flip h, 180° rotation r



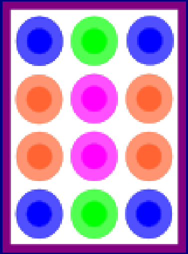
- Note that $v^2 = h^2 = r^2 = e$ and $vh = hv = r$.
- Reinforces visualization and introduces symmetry groups

Orbits: The meaning behind the colors

Let $G = \{e, h, v, r\}$ be the group of transformations, and label the positions:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

The orbit is defined by $orb_G(s) = \{\phi(s) | \phi \in G\}$.



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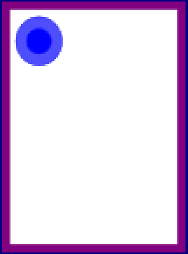

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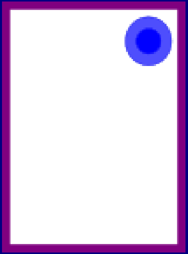

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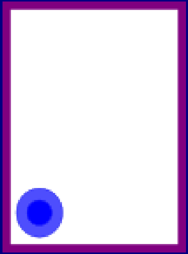

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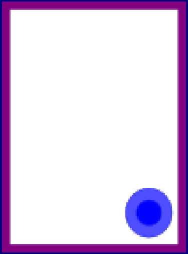

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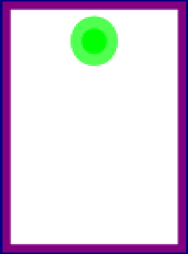

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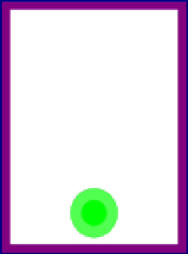

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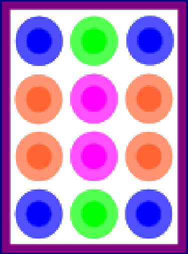

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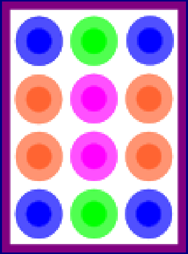

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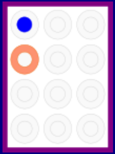
Note that green and purple symbols are fixed by a horizontal flip.

How many distinct cards are in a Swish deck?

- There are $12 \cdot 11 = 132$ ball-hoop configurations.
- Two cards are equivalent if and only if they are in the same orbit.



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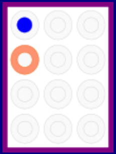

Define *fix*(ϕ) to be the set of ball-hoop configurations that are fixed when acted on by ϕ, for $\phi \in G = \{e, h, v, r\}$.

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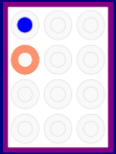

Define $fix(\phi)$ to be the set of ball-hoop configurations that are fixed when acted on by ϕ, for $\phi \in G = \{e, h, v, r\}$.

Burnside's Theorem

The number of orbits is

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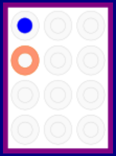

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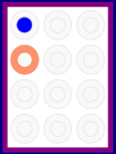

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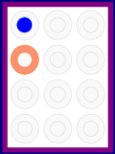

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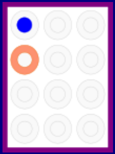

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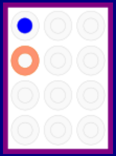

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$$= \frac{1}{4}\left(132 + 12 + 0 + 0\right) = 36$$

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There are 36 distinct cards in a Swish deck.

Probability: Not all colors are created equally!

Colors are distributed as follows:

	Blue ball	Orange ball	Purple ball	Green ball	
Blue hoop	6	6	3	3	18
Orange hoop	6	6	3	3	18
Purple hoop	3	3	2	4	12
Green hoop	3	3	4	2	12
	18	18	12	12	60

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- What is the probability that a card contains at least one blue symbol?
- What is the probability that a card contains a blue symbol and a orange symbol?
- If a card contains a blue hoop, what is the probability it contains an orange ball? Are these events independent?

What is the probability of a 2-Swish?

- A Swish deck contains 60 cards total.
- Exactly 46 of the possible $\binom{60}{2}$ pairs are swishes.
- If two cards are randomly selected:

$$P(2 \text{ swish}) = \frac{46}{\binom{60}{2}} = \frac{23}{885} \approx 0.026$$

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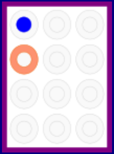
- In 16 cards, the expected number of 2-swishes is

$$\binom{16}{2} \cdot \frac{23}{885} \approx 3.12$$

Connections to linear algebra

- Each card corresponds to a matrix: 1 for a ball, -1 for a hoop, 0 else



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- Sets of matrices which sum to 0 correspond to swishes.

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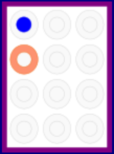

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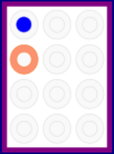

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- Theorem: A swish can be formed using any card together with a subset of the following six cards:

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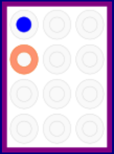

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Any vector can be written as a linear combination of basis vectors.

Summary

Swish provides a fun way to explore mathematical ideas, including:

- Reinforcing transformational geometry
- Introducing symmetry groups
- Identifying patterns
- Making conjectures
- Illustrating Orbit-Stabilizer and Burnside's Theorems
- Calculating discrete probabilities
- Connecting to linear algebra concepts

Opportunities for further explorations:

- How can the game to generalized
  - To higher dimensions?
  - To differently shaped cards?
- How can the game be simulated efficiently?
- How many different k-card Swishes contain a given card?
- In an initial layout of 16 cards,
  - What is the probability of a k-card Swish, for $3 \leq k \leq 12$?
  - What is the probability there are no swishes with three or more cards?
  - What is the expected number of swishes?
- ... and more!

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