USING TOOLS TO COMMUNICATE IN A MATH TEACHERS' CIRCLE

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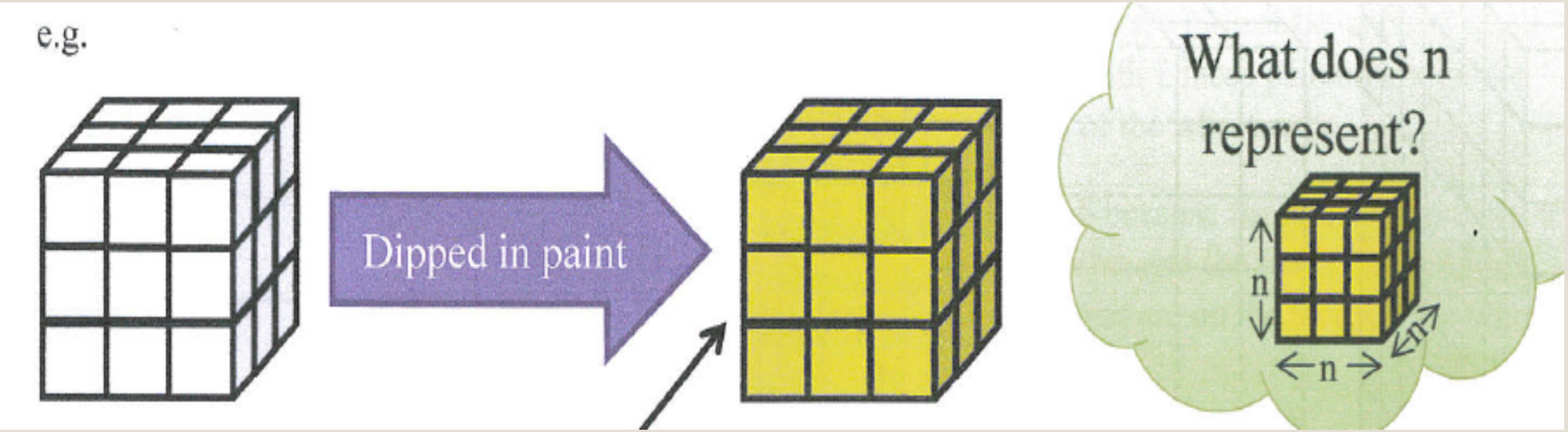
Ever Popular *Painted Cube Problem*

(nrich.maths.org)

<u>Math Objectives</u>:

- Explore constant, linear, quadratic, and cubic functions.
- Develop, test, and justify general rules.
- Construct viable arguments (i.e. mathematical proof).
- Use multiple representations to express solution (technology, patterns, finite differences, geometrical representations, etc.)

Problem: A white cube made up of smaller white cubes is dipped into a bucket of yellow paint, resulting in the surface area of the cube being painted yellow. How many faces (0-6) of the cubes are covered in yellow paint? How many faces of the cubes will be covered in yellow paint for any size n x n x n cube?

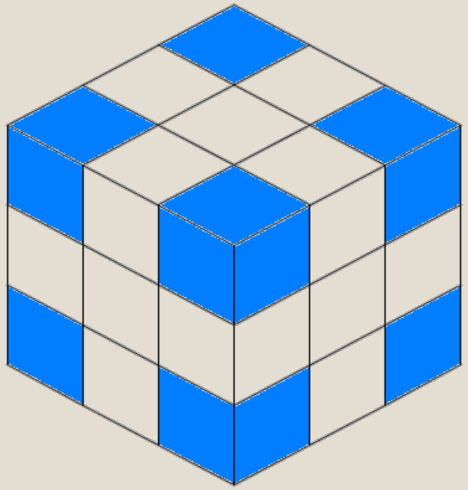


To simulate the painted cube, use small cubes to make the larger cube. Place dot stickers on the cube to simulate the paint.

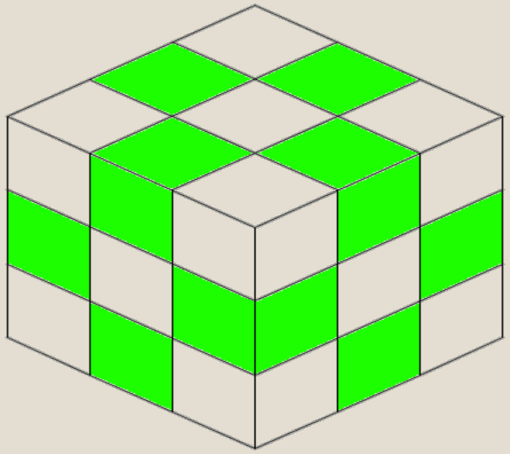
Assumptions and what we already know:

- When the cube is dipped into the paint, the paint only covers the surface area of the cube.
- We are counting the number of smaller cubes within the whole cube.
- The whole cube is completely made up of small cubes; there are no gaps or holes.
- On a whole cube there are 8 corners
- On a whole cube there are 6 faces
- On a whole cube there are 12 edges

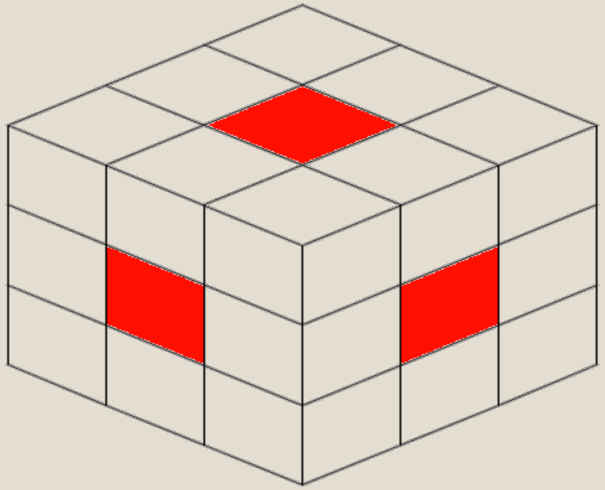
3 x 3 x 3 Cube Sample



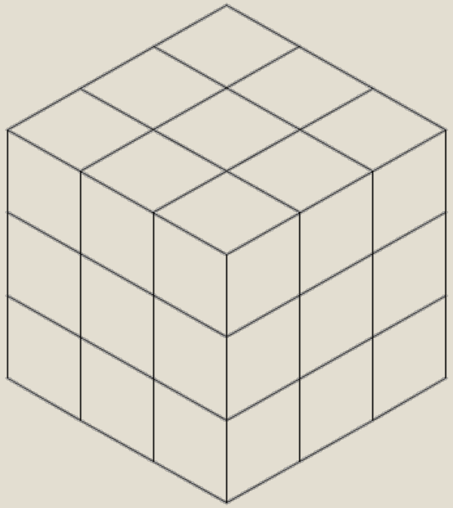
3 faces painted



2 faces painted



1 face painted



0 faces painted

Strategy: Create a Simpler Problem

- Consider a 2 x 2 x 2 cube.
- How many unit cubes does it take to build a 2 x 2 x 2 cube?
- If the 2 x 2 x 2 cube was dipped in paint, what is the greatest number of faces of a single unit cube that could be painted?
- How many faces of each of the unit cubes are painted on the 2 x 2 x 2 cube?

Record Conjectures

Conjecture 1

Additional Details

In order to prove or disprove this conjecture I will need to:

This conjecture turned out to be (circle one):

Incorrect Partially correct Correct

MTC Participants:

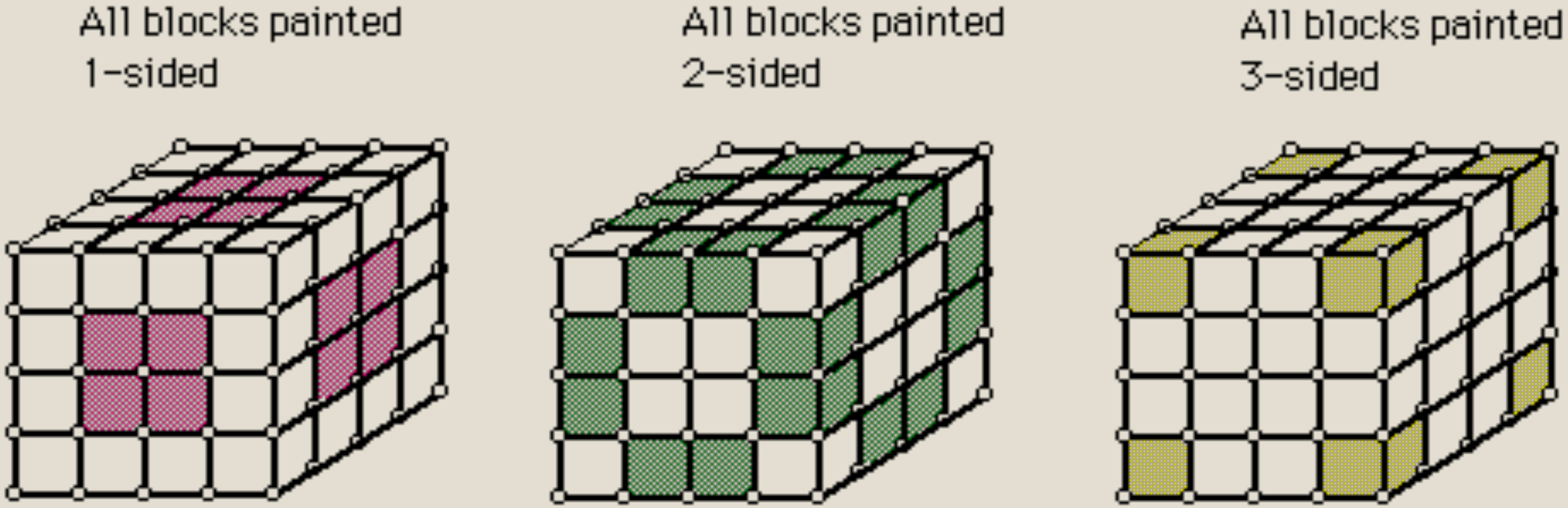
- Built cubes of varying sizes (i.e. 2x2x2, 3x3x3, 4x4x4, 5x5x5, etc.).
- Discussed strategies for determining how many unit cubes are painted.
- Organized data in a table.
- Discussed possible patterns in the table.
- Determined general expressions for the cube of size n.

Dimensions	Number of 1 x 1 x 1 cubes needed	3 faces painted	2 faces painted	1 face painted	0 faces painted
2 x 2 x 2					
3 x 3 x 3					
4 x 4 x 4					
5 x 5 x 5					
.					
.					
n x n x n					

Organize Findings in a Table

n (side length of cube)	Number of unit cubes with paint on zero faces	Number of unit cubes with paint on one face	Number of unit cubes with paint on two faces	Number of unit cubes with paint on three faces
2	0	0	0	8
3	1	6	12	8
4	8	24	24	8
5	27	54	36	8

4 x 4 x 4 Cube Sample



Patterns

Sample Answer: For any value of n where $n \geq 2$, there will always be 8 cubes that have three faces painted.

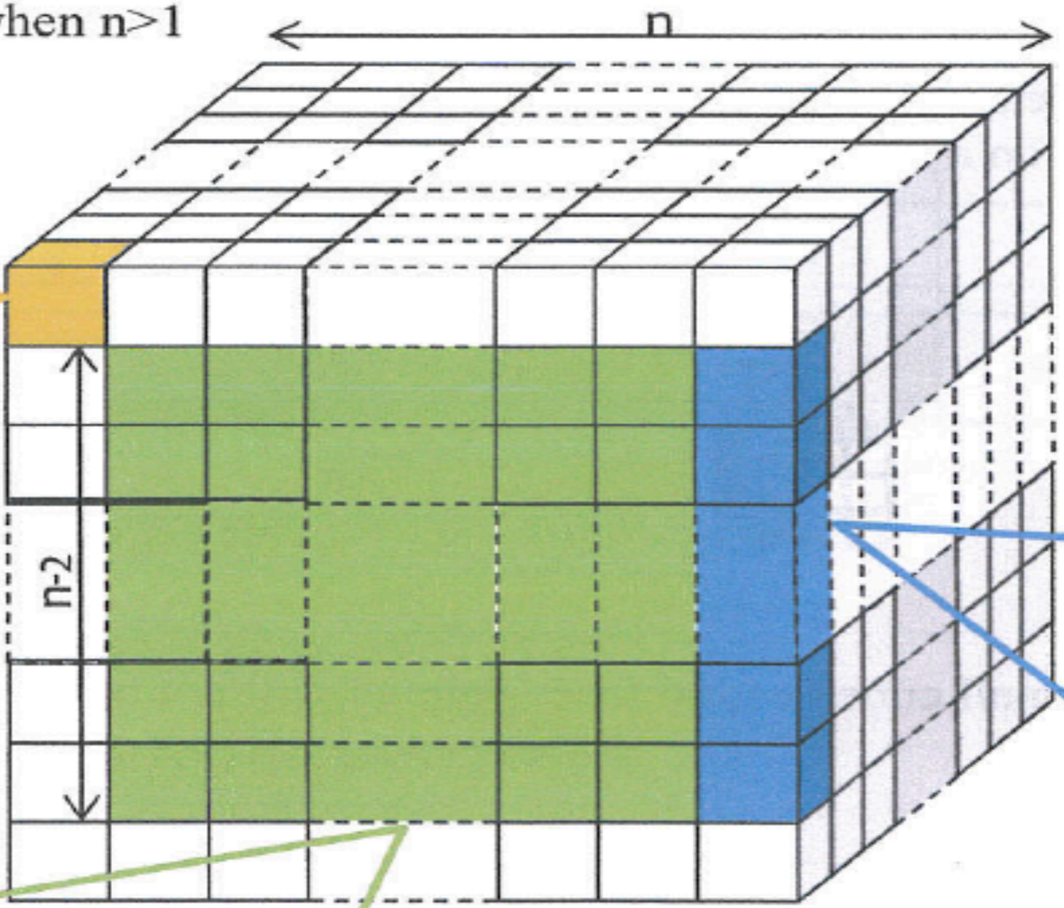
The number of cubes with two faces painted is always a multiple of 12. The number of unit cubes with paint on two faces increases by 12 each time.

For the number of cubes with one face painted, the number increases by 6, then 18, then 30. There is a difference of 12 between each of those numbers.

The number of unit cubes with paint on zero faces is a perfect cube. It is the cube of the number that is two less than the side length of the cube.

The formulas:

The Formulas only work when $n>1$



There is 1 small cube with three faces painted for each corner of the whole cube.

There are (n-2) small cubes with 2 faces painted for each edge of the whole cube. This is because 2 small cubes on each edge are corners, which have 3 of their faces painted. So, to get the number of small cubes on an edge that have 2 faces painted you must use the length of one edge (this dimension = n) and subtract 2.

There are $(n-2)^2$ small cubes with one face painted for each face of the whole cube. This is because on each face of the whole cube the small cubes round the edge either have 2 or 3 faces painted. This leaves a square of small cubes in the middle of each whole cube face which only have one face painted. To get the height of this square you must use the height of one face of the whole cube (this dimension = n) but subtract 2 from it to get rid of the edges and corners. All you then have to do is square it to get the number of small cubes in this middle square.

Number of small cubes	Number of small cube faces painted							Number of small cubes in the whole cube	
	6	5	4	3	2	1	0		
1	1	0	0	0	0	0	0	$= 1$	1^3
2	0	0	0	8	0	0	0	$= 8$	2^3
3	0	0	0	8	12	6	1	$= 8+12+6+1 = 27$	3^3
n	**0**	**0**	**0**	**8**	**$12(n-2)$**	**$6(n-2)^2$**	**$(n-2)^3$**	$8+12(n-2)+6(n-2)^2+(n-2)^3 =$	n^3

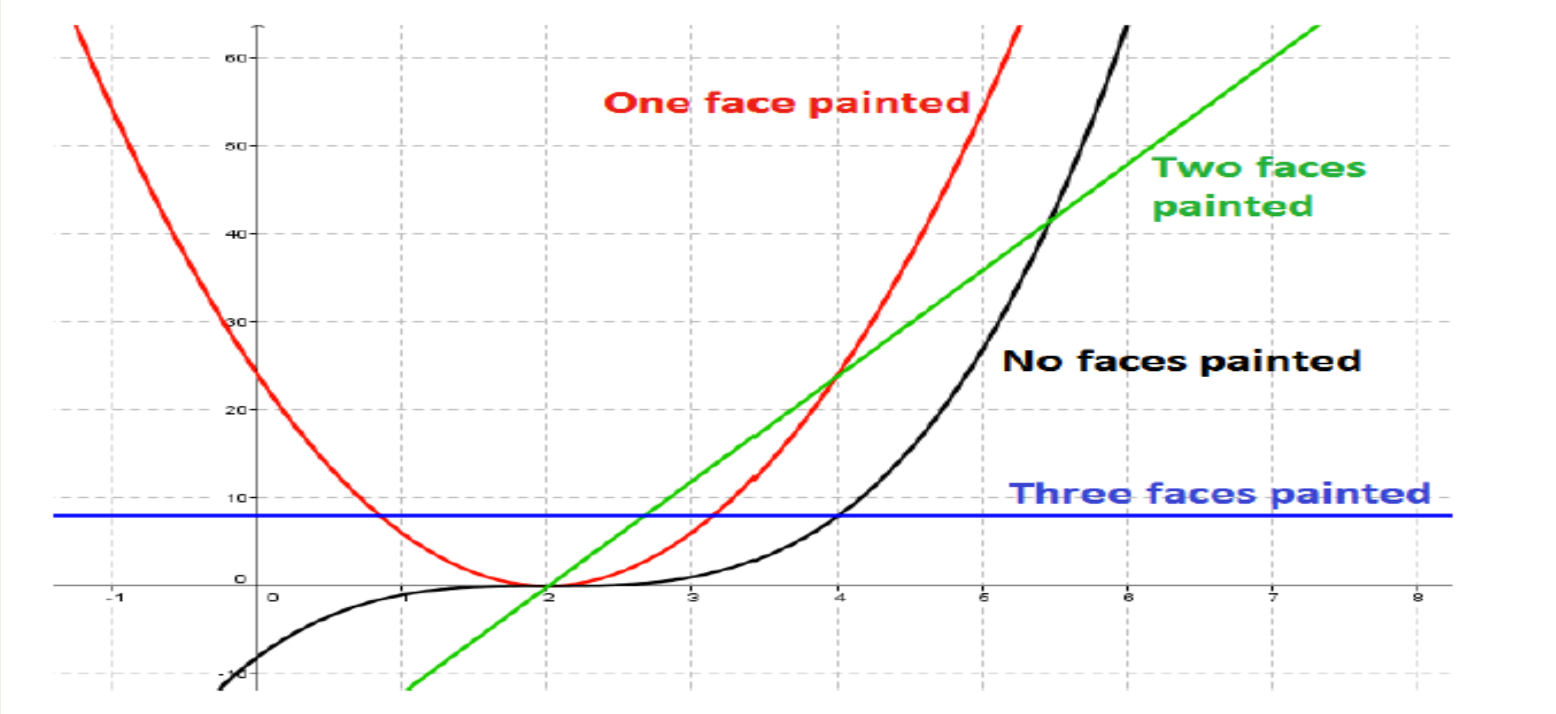
Additional Conclusions

- It is impossible for any cube to have 4 or 5 painted faces.

- For any cube bigger than a 1 x 1 x 1, it is impossible to have a cube with 6 painted faces.

Graphs of Painted Cubes

Since the functions are successively constant, linear, quadratic and cubic, the graphs, in the long run, will show why non-painted cubes become more common than any of the others:



Painted Cube: A Probability Extension

You are given an n x n x n cube. All of the faces of the n x n x n cube are painted and then cut into 1 x 1 x 1 cubes. You take all of the 1 x 1 x 1 cubes, put them in a container, shake it up, and then randomly draw out one of the cubes. You roll the cube like a die. What is the probability that the top face of the cube would be painted?

Cube Dimension	Total # cubes	Total # of Faces	Total # of Painted Faces	P(rolling a painted face)
2 x 2 x 2	8	48	24	½
3 x 3 x 3	27	162	54	1/3
4 x 4 x 4	64	384	96	1/4
…				
n x n x n	n^3	$6n^3$	$6n^2$	1/n

Conversation