The Number Machine

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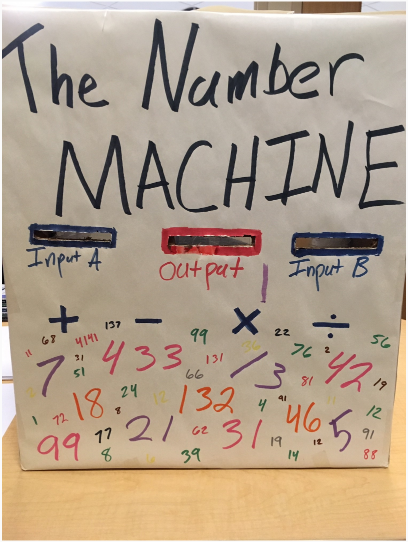

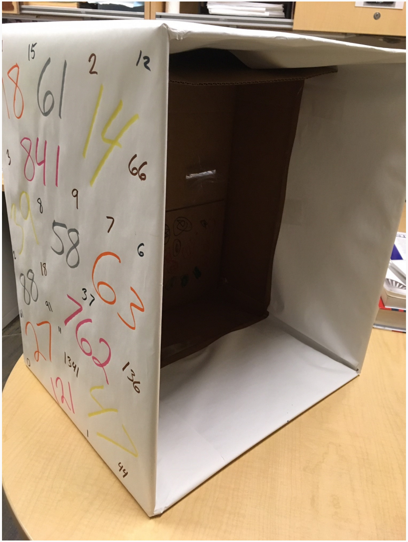

Guess the Pattern

- Students are given a set of index cards with numbers written on them.
- They put in two cards and one card comes out.
- Repeat until they can figure out the pattern/formula.

I love how the “Number Machine” makes concrete the abstract idea of a function. It aslo makes for infinite possibilities for patterns given inputs x and y:

1. $x + y$
2. maximum of x and y
3. $x \cdot y$
4. $2x + y$
5. $x + y + xy$
6. $\vdots$

Enter the Spreadsheet

The Number Machine

File Edit View Insert Format Data Tools Add-ons

100% $ % .0 .00 123 Arial

fx =if(B2="", "", A2*B2+A2+B2)

	A	B	C
1	**Input 1**	**Input 2**	**Output**
2	7	5	47
3	2	2	8
4	3	1	7
5	0	1	1
6	1	0	1
7	0	2	2
8	2	0	2
9	1	3	7
10	5	7	47
11	3	5	23
12	4	6	34
13	10	10	120
14			
15			

Question

Does any recognize the formula $x + y + xy = (x + 1)(y + 1) - 1$*?*

This formula is the basis for another classic problem that can be implemented on the number machine.

A Classic Problem

Question

I will give you index cards with the numbers $1, 2, 3, \ldots, 10$ *on them, and you will feed them into the machine two at a time, taking the output card and using it along with the others until you only have one card left.*

- *How many times will you feed cards into the machine?*
- *How should you feed the cards in so that the number in your hand at the end is the largest possible? Smallest possible?*
- *What if you had more or fewer cards?*
- *What if you had the numbers from 1 to 100?*

The Source

An Interesting Way to Combine Numbers

Joshua Zucker and Tom Davis
October 12, 2016

Abstract

This exercise can be used for middle school students and older. The original problem seems almost impossibly difficult, but there are obviously many ways to approach it by considering simpler problems. As they investigate it, students are actually drilling their multiplication and addition facts. In addition, depending on how detailed a study you want to make, the students will be forced to do a lot of simple algebra to obtain the results they need. They may also learn something about commutativity, associativity, and symmetric functions.

1 Introduction

This document is meant for the teacher. It describes an interesting problem and then lists a number of ways the problem can be used in a classroom. Depending on the age and sophistication of the students, the classroom discussion can be taken in different directions.

As usual, if you are the teacher, you will probably find this document more useful if, before reading our discussion about classroom presentation, you try to make some headway on the problem yourself. It will help you to "think like a student" and you may come up with additional ideas and strategies that did not occur to the authors.

The 1 to 100 Problem

2 The Problem

Imagine that all the numbers from 1 to 100 inclusive are written on the blackboard. At every stage, you are allowed to erase two numbers that appear on the board (let's call the numbers you erased x and y) and in place of the two erased numbers, write the number $x+y+xy$. Repeat this operation until only a single number remains.

What are the possible values for that remaining number?

Back to the Number Machine

It's a Process

The concrete nature of the Number Machine and the cards makes this deeper question easier to internalize for younger students who can see the cards go in and be consumed until there's only one card left.

- Now we want the pattern for $f(\{1, 2, 3, ..., n\}) =?$
- That's a big jump in abstraction.
- Computation, by hand then spreadsheet.
- Finding the pattern.
- Elucidating the pattern.

Solve a Simpler Problem

Question

What if the number machine did $f(x, y) = xy$? What would happen then?

Solve a Simpler Problem

Question

What if the number machine did $f(x, y) = xy$? What would happen then?

You would end up with $1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ after you fed in the numbers $1, 2, 3, \ldots n$. It's not hard to see why.

Upgrade

Applying this logic works with
$f(x, y) = x + y + xy = (x + 1)(y + 1) - 1.$

$$
\begin{aligned}
f(1, 2) &= 2 \cdot 3 - 1 \\
f(2 \cdot 3 - 1, 3) &= (2 \cdot 3 - 1 + 1)(4) - 1 = 2 \cdot 3 \cdot 4 - 1 \\
f(2 \cdot 3 \cdot 4 - 1, 4) &= (2 \cdot 3 \cdot 4 - 1 + 1)(5) - 1 = 2 \cdot 3 \cdot 4 \cdot 5 - 1 \\
&\vdots
\end{aligned}
$$

After feeding through $1, 2, 3, ..., n$ you get $(n + 1)! - 1$.

Conjecture

Question

What about other number machine patterns? If the number machine is $f(x, y) =$??? what is the end result after feeding through the numbers $\{1, 2, 3, ..., n\}$? If...

- $f(x, y) = x - y$
- $f(x, y) = 2x + y$ *(Does order matter now?)*
- $f(x, y) = x^2 + y^2$
- $f(x, y) = x^y$
- $f(x, y) = xy + 2x - y$
- $\vdots$

Now the students (or teachers) have a chance to play, to try things out, it's really open ended.

The End is Nigh...

Thanks For Your Attention!

Contact me at Tom.Clark@dordt.edu.

References

1. Joshua Zucker and Tom Davis. "An Interesting Way to Combine Numbers." `http://www.geometer.org/mathcircles/numbercombine.pdf`