Winner's Curse

MathFest (Games in Math Circles) Ted Alper
August 6, 2021 tmalper@stanford.edu

Playing "Lines of Action" rules from the great Sid Sackson book "A Gamut of Games," Dover 2011 reprint.(You can also find the rules here: https://en.wikipedia.org/wiki/Lines_of_Action.) It's Peet's Tea's turn to move, but Diet Snapple has a certain victory.

About the Stanford Math Circle

- Founded 2005, expanded repeatedly.
- three quarters per year, 10 sessions per quarter.
- moved online last year, (mostly successfully!), online for foreseeable future

Grades 9–12	2 sections	Now 40 students each	guest instructors
Grades 7–8	2 sections	30 students each	guest instructors
Grades 5–6	3 (2020) sections	25 students each	same instructors all year
	5 (2021) sections		
Grades 3–4	7 sections	at most 25 students each	same instructors all year
Grades 1–2	3 sections	at most 25 students each	same instructors all year

I lead sessions myself if I can't fill a date. Consequently, with 120 slots to fill, I am always looking for new sessions leaders.

(Also looking for an elementary math circle instructor for January-June 2022, and possibly beyond.)

Confessions

Is this *really* a game? It's an activity.

Anyway, I'm happy to share this topic, but my real motive for presenting this is to get your opinions about it, especially when running it online.

I'm including condensed versions of questions I might ask students, as well as "meta-questions" about how it runs that I'm asking you.

But let's get to the idea:

A jar of bottlecaps

How much would you bid to win a prize worth (in dollars) the number of bottlecaps in this jar? https://forms.gle/MzdWSZkshKKWtE7t7

Questions before you bid: As a bidder, how do you "win" such an auction? What does losing mean? Are there other outcomes? How do you maximize your chances of "winning"? Do you even want to do that?

(Meta-questions: what could go wrong when running such an example? What friendly rewards and consequences can encourage students to want to be the high bidder, yet be risk-averse?)

And what questions might we ask when we see the results?

Oh, why not?

A classic puzzle, slightly off-topic, yet worth repeating.

Please guess an integer between 0 and 1000. The winner will be the person whose guess is closest to 50% of the average (mean) of all the guesses.

`https://forms.gle/mTrNUYbAb6NTeNdFA`

(Just between us: what is *supposed* to happen? What do you think will actually happen?)

(Same meta-questions: what could go wrong when running such an example? What rewards or consequences can encourage students to both want to win the auction, yet be risk-averse?)

OK, back to the context

The phenomenon we're exploring is this:

- There's an object with a fixed, but unknown true value V that is being sold off.
- Each individual has some way of estimating V. (What might happen in real life? What might happen in a game?)
- Everyone bids (How do they bid? Does it matter?)

This is a real life problem, and applies most famously to oil and mining leases, auctions of portion of the electromagnetic spectrum, the construction industry, and internet advertising. Billions of dollars were lost for want of understanding this issue. Nobel prizes have been awarded for results that build upon this.

Why I like these problems for a math circle

- It's a [simplified version of a] real-life problem, with profound implications in major industries.
- A chunk of the analysis is accessible to middle-schoolers.
- It introduces a "paradox" of conditional probability with a strategic consequence that is initially counterintuitive
- We can spend more or less time discussing qualitative ideas of "fairness" and symmetry and auction design.
- It has natural connections to a lot of common math circle topics.
- (in-person) The mechanics of an auction and tabulating results can be a fun activity
- opportunities to work in small and large groups.
- (in-person) Everyone likes polyhedral dice.

Why I don't

- It's a game with few winners. Sometimes very few. Sometimes *none*.
- It can seem unfair if you repeatedly receive very low or very high estimates and feel shut out.
- The games can have fiddly rules and goals that are easy to misunderstand
- Just a few students misunderstanding can distort the intended effect.
- I struggle with providing suitable incentives/disincentives, especially online, but also in person.
- I'd like to repeat the auctions to show a learning curve and to increase the number of times each player wins, but too many auctions becomes repetitive.
- Fun polyhedral dice can be a distraction.

Game 1 (two player, in-person only)

Each player secretly rolls an N-sided die and records that roll, along with his or her secret bid (the number of points they are willing to pay to receive the value of the sum of the two die rolls in points). Players compare their rolls and bids; the lower bidder's score is unchanged, the higher bidder gains or losed an amount equal to the difference between their bid and the sum of the two dice. (If it's a tie, each player gains or losed half that difference).

It's more fun with 10-sided and 20-sided dice, but it's easier to have students compare explicit strategies with only 6-sided dice (making a 6×6 grid).

Players should try to maximize their own score, not necessarily to beat their opponent.

Meta-question: What can go wrong? How to make it work?

One idea for when [if] we return to in-person circles: have students play round-robin.

Game 2, in-person version

The entire class, then small groups, then the entire class again. A variety of prizes, each worth somewhere between 200 and 3000 "dollars". Each player gets an estimate of the value of the item which is chosen to be the true value plus or minus an error, which is (uniformly) chosen to be between 100 dollars below the true value and 100 dollars above it. Each player then secretly bids on the device. What should you bid?

When do you win? When do you lose? Who is your opponent?

When doing this **in-person**, I print out "cards" with the random values. I tried a separate deck for each auction, with three different auctions , but it's a little less fiddly to make a single deck, in which each card had values for five rounds.

SAVE THIS CARD AND KEEP THE VALUES HIDDEN!
WE WILL USE IT THROUGHOUT THE SESSION

Auction	Your Estimate:
A	2261
B	2762
C	1314
D	2312
E	586

Game 2, online version

Online. I was going to try to run this online this past year, but managed to fill enough slots that I didn't have to. It's still in reserve for this coming year.

While I'm still struggling with how to adapt epistemic-logic puzzles for online classrooms, I have figured out a way to do this. I can use a mail merge to email students their individual estimates, and a link to a google form for them to enter their bids. Some will be in groups of 3 to 5, others will be for the entire section.

For today, I've jury-rigged a version of this. If you want to try playing right now, go to: https://forms.gle/tzUj24aSaWEXZn2v5

You'll get an email with estimates for two auctions

auction A You'll recieve the true value plus an error uniformly chosen between -200 and 200.

auction B You'll receive the true value multiplied by a real number uniformly chosen between 0.5 and 1.5

as well as links to forms to submit bids for each auction.

Extremely Brief Analysis

The *expected* phenomenon: average bids are probably a bit low, but winning bids are too high. And the more people who participate in the auction, the more dramatic the effect.

- A natural intution is that the probability the true value is above your estimate is about 50%.

- But what is the true value likely to be on the occasions when everyone else's estimate is lower than yours?

We don't have to use formal notation for conditional probability or expected value (though with some middle school audiences, we can!). And the computations are not in general tractable for middle-schoolers though we can break them up into *somewhat* accessible steps, especially for small values of the relevant parameters, and then display some simulations.

For example, with k random samples of integers chosen uniformly between -200 and 200, the expected value of the highest value is about $200\frac{k-1}{k+1}$.

1	0	6	143.21	11	167.08	16	176.91
2	66.83	7	150.37	12	169.65	17	178.22
3	100.25	8	155.94	13	171.85	18	179.39
4	120.30	9	160.40	14	173.76	19	180.45
5	133.67	10	164.04	15	175.43	20	181.40

What does this suggest our strategy should be?

What about collusion, either by sharing estimates or lowering bids?

Should we just avoid all such auctions?

If we're the seller at the auction, what could go wrong for us? What could we do?

other connections

- knowledge about knowledge or epistemic logic puzzles (for example, holding cards up to your head so you can see red and black cards on everyone else's head but not your own) – meta question: how do you do *that* online? Many many variants, and I'd like to find a painless way to simulate what is so easy to do in person.

- Game theory, nash equilibrium

- other probability and conditional probability puzzles, Bayes' theorem (which we implicitly use here, but sidestepped). Paradoxes like two-envelopes problem

- other auction theory (Vickrey auctions have come up at the high school math circle), or economics topics (stable matching)

Further reading, more details

http://veconlab.econ.virginia.edu/cv/cv.php
links to a college economics lab where you can create an account to run similar experiments

"The Winner's Curse" is a readable economics book for a general audience by Richard Thaler (also a Nobel laureate), that covers many seemingly irrational economic behaviors. The entire book is worth reading, but the title essay exploring just this phenomenon is available independently.

And of course, see the press release and more detailed info on the 2020 Nobel Prize in Economics for Paul Milgrom and Robert Wilson.

15