The wrangle at JMM 2017 was snowed out!
So these problems were saved for Math Fest

# JMM 2017 Math Wrangle Solutions

**1** *Let $g(x) = e^x + e^{-x}$. Define real numbers $a$ and $k$ so that $g(a) = k$. Determine $k$ so that for each natural number $n$, $g(na)$ is constant.*

**Solution:** If it is always the case that $g(na) = e^{na} + e^{-na} = k$, then

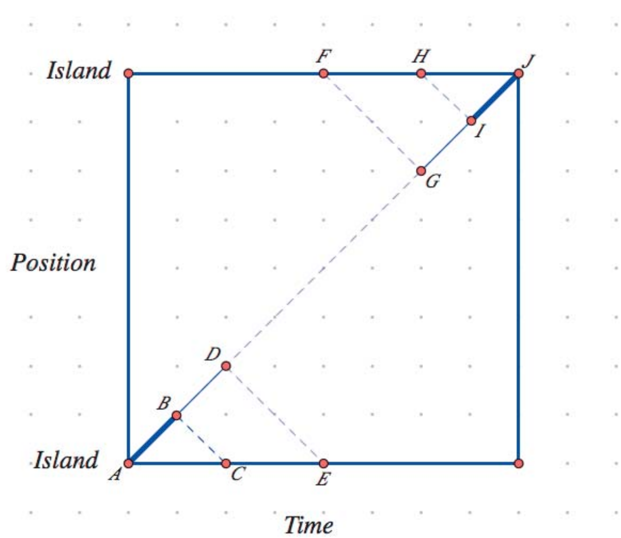
$$k^2 = g(a)g(na) = \left(e^a + e^{-a}\right)\left(e^{na} + e^{-na}\right) = e^{(n+1)a} + e^{-(n+1)a} + e^{(n-1)a} + e^{-(n-1)a}$$

$$= g(na) + g((n-1)a).$$

If $n = 1$, this implies $k^2 = k + 2$. For $n > 1$, we get $k^2 = 2k$. The only possibilities are $k = 0, -1, 2$, but since exponentials are always positive, $k = 2$ is the only solution.

**2** *A group of airplanes is based on a small island. The tank of each plane holds just enough fuel to take it halfway around the world. Any desired amount of fuel can be transferred from the tank of one plane to the tank of another while the planes are in flight. The only source of fuel is on the island, and we assume that there is no time lost in refueling either in the air or on the ground. What is the smallest number of planes that will ensure the flight of one plane around the world on a great circle, assuming that the planes have the same constant ground speed and rate of fuel consumption and that all planes return safely to the island base?*

**Solution:** Three planes can accomplish this. The following distance-time graph explains it. The horizontal axis is time, divided into 8 units, where each plane can fly for 4 units of time. The vertical axis is also in 8 units, but note that the top and bottom are the same position—the island—and that each plane can travel for 4 units (halfway around the world). Here is the algorithm. Call the planes 1, 2, 3. We will get plane 1 around the world, using the other planes for refueling.



**A** Planes 1, 2, 3 take off together from island.

**B** Plane 3 gives one unit of fuel to each of 2 and 1. So 1 and 2 now have full tanks, and 3 has one unit left.

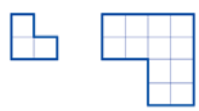
**C** Plane 3 returns to island to refuel and wait.

**D** Plane 2 gives one unit to plane 1, and heads back to island (**E**) to refuel. Now plane 1 has a full tank, and can travel all the way to **G**.

**F–J** Planes 2 and 3 do what they did earlier, but in reverse. As soon as 2 arrives at **E**, it refuels and takes off, to meet plane 1 at **G** and give it one unit of fuel. Meanwhile, plane 3 leaves the island at **H**, arriving at **I** just in time to give both 1 and 2 each one unit of fuel. Now all three planes have one unit left, enough to all get back to the island at **J**.

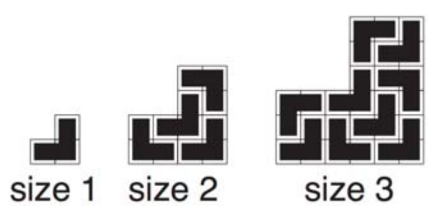
Question: You still need to show that two planes will not suffice! How can you argue this?

**3** *Define a* size-$n$ tromino *to be the shape you get when you remove one quadrant from a* $2n \times 2n$ *square. In the figure below, a size-1 tromino is on the left and a size-2 tromino is on the right.*



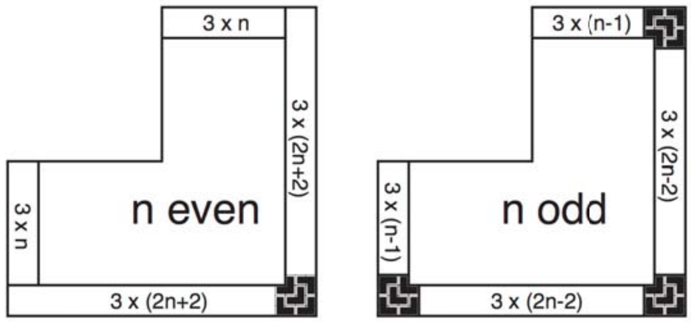
*We say that a shape can be* tiled with size-1 trominos *if we can cover the entire area of the shape*—and no excess area—*with* non-overlapping *size-1 trominos. For example, it is easy to see that a* $2 \times 3$ *rectangle can be tiled with size-1 trominos, but a* $3 \times 3$ *square cannot be tiled with size-1 trominos. Can a size-2017 tromino be tiled by size-1 trominos?*

**Solution:** For any $n$, we can tile an size-$n$ tromino with size-1 trominos. We can prove this by induction. First, note that you can do this for $n = 1, 2, 3$:



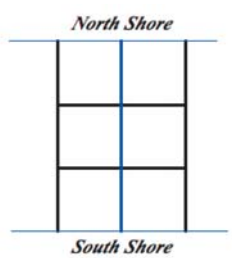

Next, note that if $k$ is even, then it is possible to tile any $3 \times k$ rectangle with size-1 trominos:

Finally, we show that it is possible, giving a tiling of a size-$n$ tromino, to go to size-$(n+3)$. There are two cases, depending on whether $n$ is even or odd.




size n+3 from size n

**4** *A system of 13 bridges, shown below, connects the north shore of a river to the south shore. For each bridge, there is a 50% probability that a protest march will block traffic across that bridge, and these probabilities are independent (imagine that each bridge flips a coin). What is the probability that it is possible to cross from one shore to the other?*




**Solution:** Look at a "dual" problem: imagine that the bridges are each drawbridges, and there is a boat waiting to the west of the bridges, and it is too tall to pass under a bridge unless the drawbridge is open. We would like to get the boat to get past all the bridges so that it gets to the east of the bridges. Suppose the probability that any drawbridge is open is $1/2$. Draw a picture, and you will see that the probability that the boat can pass from west to east is equivalent to the original problem! But if the boat can pass from east to west, then it is impossible to travel from north to south, and conversely, if it is possible to travel from north to south, the boat cannot pass from west to east. In other words, if we substitute "open drawbridge" for "protest march" we see that whenever the bridges forbid north-south pedestrian travel, they allow west-east boat travel, and conversely, whenever the bridges allow north-south pedestrian travel, boats cannot go west-east. So the probabilities are both equal to $1/2$!

**5** *Consider a deck of cards with the numbers #1 through #20 on them. 3 cards are picked at random. Find the probability that at least one of the following is true of the hand chosen:*

**(A)** *All 3 Cards are odd.*

**(B)** *All 3 Cards are prime numbers.*

**(C)** *None of the cards is divisible by 3.*

**Solution:** Let's first determine the sizes of the underlying sets and their intersections:

- $A = \{1,3,5,7,9,11,13,15,17,19\} \implies |A| = 10 \implies \binom{10}{3} = 120$
- $B = \{2,3,5,7,11,13,17,19\} \implies |B| = 8 \implies \binom{8}{3} = 56$
- $C = \{1,2,4,5,7,8,10,11,13,14,16,17,19,20\} \implies |C| = 14 \implies \binom{14}{3} = 364$
- $A \cap B = \{3,5,7,11,13,17,19\} \implies |A \cap B| = 7 \implies \binom{7}{3} = 35$
- $A \cap C = \{1,5,7,11,13,17,19\} \implies |A \cap C| = 7 \implies \binom{7}{3} = 35$
- $B \cap C = \{2,5,7,11,13,17,19\} \implies |B \cap C| = 7 \implies \binom{7}{3} = 35$
- $A \cap B \cap C = \{5,7,11,13,17,19\} \implies |A \cap B \cap C| = 6 \implies \binom{6}{3} = 20.$

The last number in each case is the number of 3 card hands in these sets. Since the total number of hands is $\binom{20}{3} = 1140$ our probability using inclusion/exclusion is

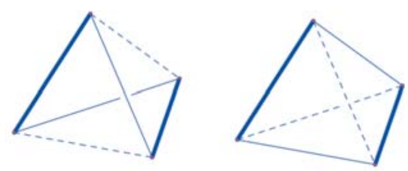
$$P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
$$= \frac{120 + 56 + 364 - 35 - 35 - 35 + 20}{1140} = \frac{455}{1140} = \frac{91}{228} \approx 0.3991$$

**6** *Colorings of the edges of two congruent regular tetrahedra are said to be* equivalent *if it is possible to perform a rotation that turns one coloring into the other. For example, if we color edges with two colors (depicted by thick or thin lines below), the two leftmost tetrahedra have equivalent colorings, but the third tetrahedron's coloring is not equivalent to the first two.*

*In how many different (non-equivalent) ways can one color the edges of a regular tetrahedron so that two edges are red, two are black, and two are green?*

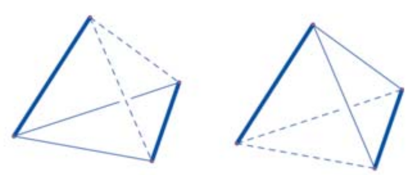
**Solution:** There are nine non-equivalent colorings where two edges are red, two are black, and two are green. In the illustrations below, we will indicate the three colors by drawing edges thick, thin, or dashed. We will partition the colorings into three cases, determined by the number of pairs of edges that are colored the same: three, one, or zero (two is impossible). For each case, without loss of generality, we fix the position of the two thick edges and carefully examine all possible colorings, and determine which are equivalent.

- *Each pair of opposite edges is colored the same.* Fixing the two thick edges, there are two possible configurations shown below.



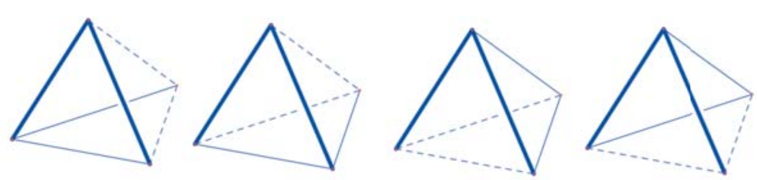
  These two colorings are non-equivalent. To see why, note that the only rotations available are 2-fold (180-degree) rotations about axes joining midpoints of opposite sides, or 3-fold (120-degree) rotations about axes joining a vertex with the center of the opposite face. The 2-fold rotations leave both pictures alone (none of them turns one picture into the other), and the 3-fold rotations do not keep the thick lines in place.

- *Exactly one pair of opposite sides is the same color.* For example, suppose that the two opposite sides are colored with thick lines. Keeping this pair in place, there are only two configurations.



  Note that these two colorings are in fact equivalent, since you can turn one into the other by performing a 2-fold rotation about the axis joining the midpoint of the two thick edges. Since there are three choices of colors we can use for the pair, this case has a total of 3 non-equivalent colorings.

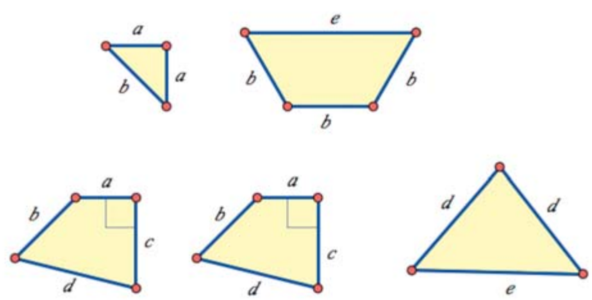
- *No pairs of opposite edges have the same color.* Again, we will fix two thick edges and carefully count the possibilities.



These are the only possible choices, once we fix the thick edges, and they are all non-equivalent, since no rotation will keep the two thick edges in place. Furthermore, the first two tetrahedra each have a thick-thick-thin face, but the first one has a face with edges colored thick-thin-dashed (going clockwise), whereas the second has a thick-thin-dashed face (going *counterclockwise*). For the next two tetrahedra, both have thick-thick-dashed faces, but the first has a thick-dashed-thin clockwise face, but the last has a thick-dashed-thin counterclockwise face.

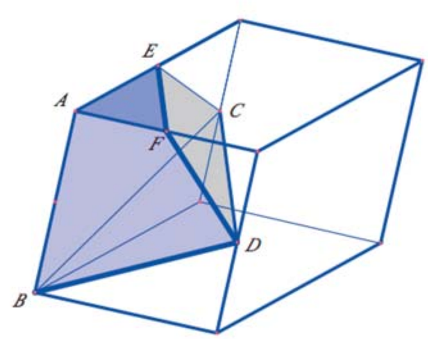
In sum, there are $2+3+4=9$ non-equivalent colorings.

**7** *Consider the five shapes below, consisting of an isosceles right triangle, an isosceles trapezoid, an isosceles triangle, and two congruent quadrilaterals that contain one right angle (marked). The lengths of these shapes are $a=1, b=\sqrt{2}, c=2, d=\sqrt{5}, e=\sqrt{8}$.*

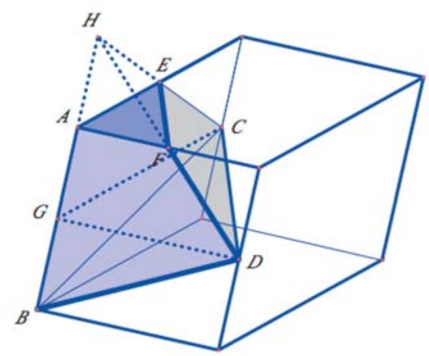



*Find the volume of the polyhedron formed when these five shapes are fitted together.*

**Solution:** The volume is $11/6$. When the polyhedron is constructed, it fits nicely in the corner of a cube with side length 2 as shown. The trapezoid is $EFDC$, with $CD$ joining the midpoints of diagonally opposite edges of the cube. The isosceles right triangle is $AFE$, lying on the top face of the cube, and $ABDF$ is one of the two congruent quadrilaterals (the other is $ECBA$). And $CBD$ is the isosceles triangle, which lies above the base the cube (sharing vertex $B$ with a vertex of the cube).



In order to compute the volume, we extend three lines and make a parallel slice to break it up into two pyramids as shown.




Point $G$ is the midpoint of $AB$; hence triangle $CGD$ is an isosceles right triangle that is parallel to the base of the cube, with area $2 \cdot 2/2 = 2$. Now we use the formula for the volume of a pyramid: $V = Bh/3$, where $B$ is the base area and $h$ is the height. Let $[\cdot]$ denote the volume of a shape. Then the volume of our polyhedron is equal to the

$$[HGDC] - [HAFE] + [BGDC].$$

The first and last of these are pyramids whose base is triangle $CDG$, with heights of 2 and 1, respectively. Hence $[HGDC] = 4/3$ and $[BGDC] = 2/3$. The middle term is a pyramid with base $AFE$ and height 1, with volume of $1/6$. Thus the volume of our polyhedron is $11/6$.

**8** *Let $F_1, F_2, F_3, \ldots$ be the* Fibonacci sequence*, the sequence of positive integers satisfying*

$$F_1 = F_2 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 1.$$

*Does there exist an $n \geq 1$ for which $F_n$ is divisible by 2017?*

**Solution:** The answer is yes. To see why, we write the sequence (mod 2017): our goal is to show that it equals zero eventually. Although conventionally, the Fibonacci sequence starts with $F_1 = F_2 = 1$, we can extend it *backwards*—this is the crux idea—with $F_0 = 0$.

Next, we can show that the sequence is eventually *periodic*: There are only 2017 different values (mod 2017), and thus $2017^2$ possible distinct consecutive pairs of numbers. By the pigeonhole principle, eventually, after at most $2017^2 + 1$ steps, we will see the same consecutive pair repeated, and this will then determine the rest of the sequence, with repeating blocks of the same numbers, *ad infinitum*.

We will be done if the periodic block begins with $F_0 = 0$, since this would imply infinitely many zeros. But perhaps the periodic block didn't start at the beginning. We will use the "extending backwards" idea to show that, in fact, periodicity must start at the beginning of the sequence ($F_0$).

Suppose that a periodic block starts at $F_M = a, F_{M+1} = b$, and has length $L$, in other words, ends at $F_{M+L-1}$, and suppose that $M > 0$. Notice that by going backwards, we can compute $F_{M+L-1} = b - a$, since the next periodic block starts at index $M + L$, and $F_{M+L} = a$ and $F_{M+L+1} = b$. Likewise, we can keep going backwards from $F_M$ to deduce that $F_{M-1} = F_{M+L-1}$ and $F_{M-2} = F_{M+L-2}$, etc., so eventually we will get $F_0 = F_{M-M} = F_{M+L-M} = F_L$. So the periodicity starts at $F_0$ and we are guaranteed to see zeros every $L$ steps.

Remark: This proof shows that the length of the period is at most $2017^2 + 1$, but in fact it is much smaller. It can be proven (using quadratic reciprocity and the Binet formula for the Fibonacci numbers) that for a prime $p$, the maximum period for divisibility (mod $p$) is $p+1$. In fact, $F_{1009} \approx 3.3040302 \times 10^{210}$ is the first Fibonacci number that is a multiple of of 2017, and every 1009th Fibonacci number thereafter will be a multiple of 2017.