Math Wrangle, JMM 2018

Problems

1 Find all solutions to $(12x-1)(6x-1)(4x-1)(3x-1)=7$.

2 How many positive numbers less than 10^9 have a digit sum of 30 (when written in base-10)? For example, one such number is 88806; another is 30009981.

3 Define w_ℓ to be the smallest n such that, if the positive integers $1,2,\ldots,n$ are colored red or black, then there must be an arithmetic progression of length ℓ, all of whose terms are the same color. For example, $w_2=3$. Find w_3.

4 A positive integer has a *trapezoidal representation* if it can be written as a sum of at least two consecutive positive integers. Define $T(n)$ to be the number of different trapezoidal representations of n. For example, $T(4)=0, T(12)=1, T(9)=2$, because 4 cannot be written as a sum of consecutive positive integers, and $12=3+4+5$ is the only way to write 12 as a sum of consecutive positive integers, but $9=2+3+4=4+5$. Find the smallest n such that $T(n)=2018$.

5 Let $\theta=2\pi/2018$. Find the value of the product

$$\cos\theta\cos(2\theta)\cos(3\theta)\cdots\cos(2017\theta).$$

6 A standard deck of cards has 52 cards, of which 4 are aces. When this deck is shuffled, what is the most likely position for the first ace?

7 Start with 2018 lengths of wire. Next, attach each end of a piece of wire to another end, choosing randomly, so that all pairings are equally likely (including attaching the two ends of a wire to one another). When this process is completed, there will be loops (no wire ends are unattached), and the number of loops will range from 1 to 2018. What is the *average* (expected) number of loops?

8 An urn contains the numbers $1,2,3,\ldots,2018$. We randomly draw, without replacement, 4 numbers in order from the urn which we will denote a,b,c,d. What is the probability that the following system will have a solution strictly inside (i.e. not on the axes) the first quadrant?

$$\begin{array}{ccccc} ax & + & by & = & ab \\ cx & + & dy & = & cd \end{array}$$