## Solutions

**1** Grouping the outer and inner terms on the left-hand side, we get

$$(36x^2 - 15x + 1)(24x^2 - 10x + 1) = 7.$$

The substitution $t = 12x^2 - 5x$ yields $(3t+1)(2t+1) = 7$, which simplifies to $6t^2 + 5t - 6 = 0$, or $(2t+3)(3t-2) = 0$, so $t = -3/2, 2/3$. Solving for $x$, we get $x = (5 \pm \sqrt{57})/24$, $(5 \pm i\sqrt{47})/24$.

**2** Initially, if we don't worry about the size of a "digit", then this is equivalent to counting the number of ordered 9-tuples $(x_1, x_2, \ldots, x_9)$ satisfying $x_1 + x_2 + \cdots + x_9 = 30$ with $0 \leq x_i$ for each $i$. With no upper bound on the variables, this is a standard ball-and-urn problem whose solution is equal to the number of ways of distributing 30 indistinguishable balls among 9 labeled urns, which equals $\binom{30+9-1}{9-1} = \binom{38}{8}$. In other words by thinking of the 30 balls laid out in a row, we have 30+8=38 balls and urn dividers. We then get a unique tuple for every choice of where the 8 urn dividers go.

From this number, we must subtract the illegal digit assignments, i.e., those for which at least one digit-urn contains at least 10 balls. Let $B_i$ denote those solutions for which $x_i \geq 10$, and let $b_i$ denote the size of $B_i$. We see that for each $i$, $b_i$ will equal the number of ways of distributing 20 balls among 9 urns (since we must start by placing 10 balls in urn # $i$), hence equals $\binom{28}{8}$. Likewise, let $B_{ij}$ denote the solutions for which both $x_i \geq 10$ and $x_j \geq 10$ for $i \neq j$. Clearly the size of this set, denoted by $b_{ij}$, will equal the number of ways of distributing 10 balls among 9 urns, which equals $\binom{18}{8}$. Finally, we note that $b_{ijk} = 1$ for any distinct $i, j, k$, since once we place 10 balls into each of three distinct urns, there are no other choices.

Using the principle of inclusion-exclusion, the number of illegal assignments is equal to

$$|B_1 \cup B_2 \cup \cdots \cup B_9| = (b_1 + b_2 + \cdots + b_9) - (b_{12} + \cdots + b_{89}) + (b_{123} + \cdots + b_{789}).$$

There are 9 terms in the first set of parentheses, and $\binom{9}{2}$ and $\binom{9}{3}$ terms in the next sets of parentheses, respectively. Hence the final answer is

$$\binom{38}{8} - 9\binom{28}{8} + \binom{9}{2}\binom{18}{8} - \binom{9}{3} = 22,505,751$$

**3** Let us indicate the colors by $A, B$. The 8-character string `AABBAABB` indicates coloring 1 and 2 with $A$, coloring 3 and 4 with $B$, etc., and since this has no monochromatic arithmetic sequences of length 3, we conclude that $w_3 > 8$.

To show that $w_3 = 9$, we look at all relevant cases, where we use the symbol `x` to mean "not yet colored." Without loss of generality, we shall start by coloring 5 with $A$. In the cases below, we attempt to avoid a monochromatic arithmetic progression of length 3, and will show that this attempt will always fail.

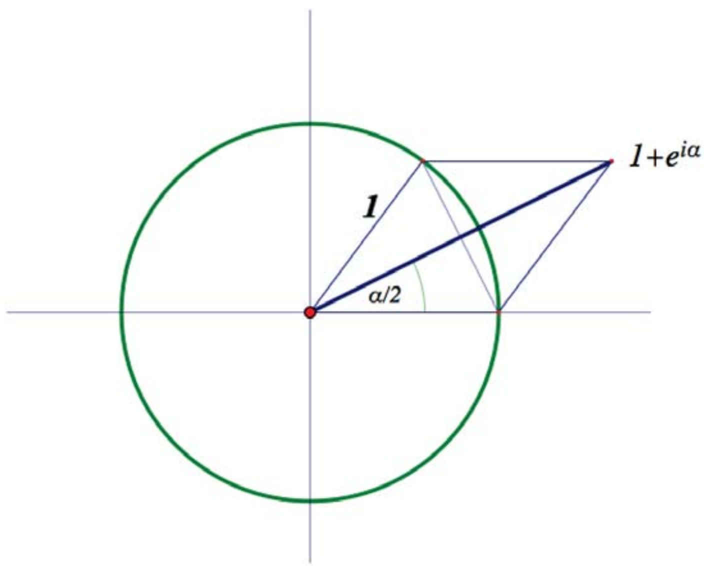
1. Positions 1 and 9 are either both $B$, or one of each color. For the latter case, by symmetry the configuration `AxxxAxxxB` suffices. This forces 3 to be colored $B$, so we have `AxBxAxxxB`, but now the $B$s at positions 3 and 9 force 6 to be colored $A$, yielding `AxBxAAxxB`. The two consecutive $A$s at 5 and 6 force 4 and 7 to be colored $B$, yielding `AxBBAABxB`, and then the two consecutive $B$s at 3 and 4 force 2 to be $A$, yielding `AABBAABxB`. Now we are done: no matter what color 8 is, we will have a 3-term arithmetic progression. (If 8 is $B$, 7,8,9 are the same color; if 8 is $A$, then 2,5,8 are the same color.)
2. The only other case, then, is to start with `BxxxAxxxB`.
   - If we color 3 $A$, we get `BxAxAxxxB` which forces positions 4 and 7 to be $B$, yielding `BxABAxBxB`, with a monochromatic arithmetic progression at positions 1, 4, 7.
   - If we color 3 $B$, we get `BxBxAxxxB` which forces `BABxAxxxB` which forces `BABxAxxBB` which forces `BABxAxABB`, and now no matter what color we choose for 6, we will get a monochromatic 3-term arithmetic progression.

**4** If $n$ can be written as a sum of $\ell$ consecutive positive integers starting at $a$, then $2n = (2a+\ell-1)\ell$. The two terms in the product are of opposite parity, both greater than 1, and the first term is strictly larger than the second. Since there is at most one trapezoidal representation of length $\ell$, and given any factorization of $2n = (2a+\ell-1)\ell$, one can solve for $a$ and $\ell$, we see that the number of trapezoidal representations of $n$ is equal to the number of different ways that we can factor $2n$ into a product of an odd and an even integer (where the odd integer is greater than 1). If we write $n = 2^r Q$, where $Q$ is odd, then any factorization of $2n$ into odd and even integers has the form $(2^{r+1}u)v$, where $uv = Q$ and $u, v > 1$. In other words, there is a one-to-one correspondence between the divisors of $Q$ greater than 1 and the factorizations of $2n$ into odd and even integers.

Thus we seek $n = 2^r Q$ where the number of divisors of $Q$ is 2019 and $n$ is as small as possible. Obviously this means we have $r = 0$. Since $2019 = 3 \cdot 673$, the smallest such $n$ is $\mathbf{5^2 \cdot 3^{672}}$.

**5** First observe that the first 1008 and the last 1008 terms of the product are the same numbers, and the 1009th term is equal to $-1$. So the entire product is negative and equal to the negative of the square of the product of the first 1008 terms.

Next we observe that $|1+e^{i\alpha}| = \pm 2\cos(\alpha/2)$, where the sign is positive when $\alpha$ is in quadrant I and negative when $\alpha$ is in quadrant II. This can be verified by a simple picture.




Thus the first 1008 terms of the product that we seek equals $|(1+z)(1+z^2)\cdots(1+z^{1008})|/2^{1008}$, where $z = e^{2\pi i/1009}$. (We know that this product is positive, because it contains an even number—504—of negative terms.)

Since $z$ is a 1009th root of unity, we can write

$$x^{1008} + x^{1007} + \cdots + x + 1 = (x-z)(x-z^2)\cdots(x-z^{1008}).$$

Plugging in $x = -1$ and observing that the left-hand side is a sum of an odd number of terms and the right-hand side is a product of an even number of terms, we get

$$(1+z)(1+z^2)\cdots(1+z^{1008}) = 1.$$

Hence the answer is $-\mathbf{1/2^{2016}}$.

**6** The most likely position for the first ace is #1; i.e., the top card in the deck. To see why, note that the probability that the first card is an ace is just $4/52$. The probability that the second card is the first ace is equal to the probability that the first card is not an ace $(48/52)$ multiplied by the probability that the second card is an ace, given that the first card is not $(4/51)$. This product is clearly smaller. In general, the probability that the first ace is at position $k$, for $2 \le k \le 49$, is equal to the probability that the first $k-1$ cards are non-aces multiplied by the probability that the $k$th card is an ace, given that the first $k-1$ are not aces. The first term in the product is equal to $C(48, k-1)/C(52, k-1)$, and the second term is equal to $4/(52-k+1)$. This equals

$$\frac{48 \cdot 47 \cdots (48-k+2)}{52 \cdot 51 \cdots (52-k+2)} \frac{4}{(52-k+1)} = \frac{4}{52}\frac{48}{51} \cdots \frac{48-k+2}{52-k+1},$$

and this is clearly a monotonically decreasing function of $k$. So the most likely location for the first ace is #1, with #49 the least likely.

**7** Let $A_n$ denote the average number of loops starting with $n$ pieces of wire. Let us consider one end of one wire. Call it "Saul." Either Saul attaches to the other end of its wire, or it does not. Since there are $2n-1$ possible ends that Saul can attach to, the probability that Saul creates a loop with its own wire is $1/(2n-1)$. In this case, the average number of loops will equal $1 + A_{n-1}$, since the remaining $n-1$ wires will form $A_{n-1}$ loops.

The other case is that Saul does not form a loop with its own wire; this has probability $1 - 1/(2n-1)$, and in this case, the average number of loops will be $A_{n-1}$, since now Saul's wire joins with another wire and we are now in the situation of pairing the endpoints of $n-1$ wires (one of these wires is longer than the others, perhaps, but that doesn't matter).

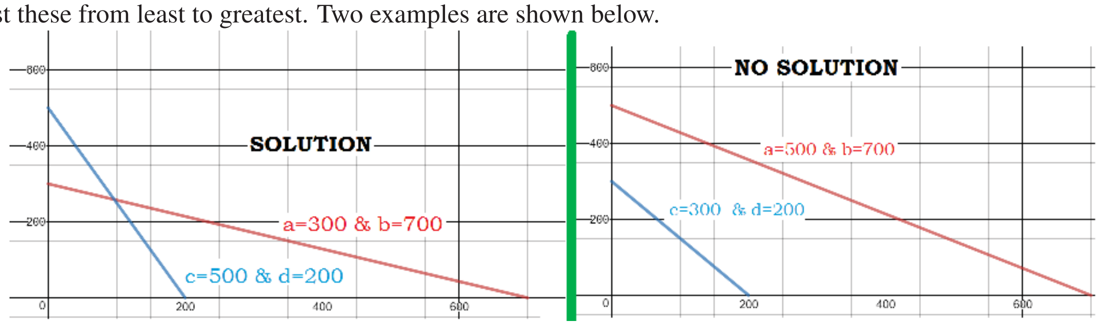
Putting these two mutually exclusive cases together yields the recurrence relation

$$A_n = \frac{1}{2n-1}(1 + A_{n-1}) + \left(1 - \frac{1}{2n-1}\right) A_{n-1} = \frac{1}{2n-1} + A_{n-1}.$$

Unpacking this recurrence, we see that

$$A_{2018} = \frac{1}{4035} + \frac{1}{4033} + \frac{1}{4031} + \cdots + \frac{1}{3} + 1 \approx 4.7867$$

**8** Equation $(1)$ is the line from $(0, a)$ to $(b, 0)$ and likewise equation $(2)$ in the line from $(0, c)$ to $(d, 0)$. Whether or not these lines meet inside the 1st quadrant is determined completely by the ordering of $a, b, c, d$ when we list these from least to greatest. Two examples are shown below.




There are $4! = 24$ ways in which these numbers can be ordered and we claim that exactly half of these give a solution. This is because the operation of switching the roles of $a$ and $c$ (which reverses where the lines meet the $y$-axis while keeping the $x$-intercepts fixed) will always move us from a situation with a solution to one without and vice-versa as illustrated above. Since each possible ordering is equally likely, the probability is $\frac{1}{2}$.