What I Learned About Sharing Long After Kindergarten

Matthew Jones

mjones@csudh.edu

CALIFORNIA STATE UNIVERSITY, DOMINGUEZ HILLS

Session Overview

- Fair sharing ideas for two or three people

- Some basic results regarding sharing

- Extended handout and PowerPoint at: https://goo.gl/eAAWgs

How can you and a partner share this cake fairly?

You have a half dozen chocolate chocolate chip cookies with walnuts, and a half dozen white chocolate cranberry oatmeal cookies. How can you share them fairly with another person?

What does it mean to share fairly?

See if you can write some criteria by which to judge whether a division of an item (or items), called an **allocation**, is fair.

Some criteria

- Proportional
- Envy-free
- Equitable
- Efficient

Proportional

An allocation among n people is proportional if each person gets a portion that is at least 1/n in that individual's valuation.

Envy-free

An allocation is envy-free if each person values his or her own portion at least as highly as every other piece.

Equitable

An allocation is equitable if each person's valuation of his or her own portion is equal to every other person's own valuation of their own portions.

Efficient

An allocation is efficient if there is no other allocation that is at least as good for everyone, and strictly better for at least one person.

Valuation for items to be divided

- Imagine each person has $100 (or 100 points) that they will "spend" on the items to be divided.

- This gives a way of measuring what each person values and of judging the allocation of items.

Some scenarios

Ana and Rob

Not to scale!!

Ana and Rob

Ana and Rob have 4 chocolate chocolate chip cookies with walnuts, 4 white chocolate cranberry oatmeal cookies, and 4 chocolate chip cookies.

Cookie Type	Ana's value	Rob's value
Choc. Choc. Chip Nuts	50	100
Wh. Choc. Chip Cran.	30	0
Choc. Chip	20	0

Ana and Rob

Assess the value received by each of Ana and Rob with the following allocation.

Cookie Type	Allocate:	
	Ana	Rob
Choc. Choc. Chip Nuts	2	2
Wh. Choc. Chip Cran.	2	2
Choc. Chip	2	2

Ana and Rob

Ana and Rob each receive a value of 50. The allocation is proportional, envy-free, equitable, and NOT efficient.

Cookie Type	Ana's value	Rob's value
Choc. Choc. Chip Nuts	25	50
Wh. Choc. Chip Cran.	15	0
Choc. Chip	10	0
Total	50	50

Ana and Rob

Assess the value received by each of Ana and Rob with the following allocation.

	Allocate:	
Cookie Type	Ana	Rob
Choc. Choc. Chip Nuts	0	4
Wh. Choc. Chip Cran.	4	0
Choc. Chip	4	0

Ana and Rob

Ana and Rob each receive a value of 50. The allocation is proportional, envy-free, NOT equitable, and efficient.

Cookie Type	Ana's value	Rob's value
Choc. Choc. Chip Nuts	0	100
Wh. Choc. Chip Cran.	30	0
Choc. Chip	20	0
Total	50	100

Rob and Carla

What are some mathematical questions about these criteria?

Mathematical questions

- Is a proportional allocation always envy-free?
- Is an envy-free allocation always proportional?
- Can you find (in our examples) envy-free allocations that are equitable?
- Can you find (in our examples) envy-free allocations that are NOT equitable?
- Can you find (in our examples) equitable allocations that are NOT envy-free?
- Is there always a (proportional, envy-free, equitable, efficient) solution?

For two people, a proportional allocation is always envy-free.

- Can you write a mathematical proof of this fact?

For any number of people, an envy-free allocation is always proportional. (Equivalently, an allocation that is not proportional must not be envy-free.)

- Write a mathematical proof of this fact.

Ana, Rob, and Carla

Mathematical questions

- Is a proportional allocation always envy-free? Yes for two people. Not necessarily for more than two people.

- Is an envy-free allocation always proportional? Yes.

- Envy-free and equitable are two independent criteria. An allocation may satisfy just one, both, or neither.

- Is there always a (proportional, envy-free, equitable, efficient) solution?

Cake for three

- Devise a way for three people to share a cake fairly.

Steinhaus lone-divider procedure for three

- A, B, and C

- A divides the cake into three pieces she finds acceptable

- B and C indicate which piece(s) they find acceptable

- Case 1: If B and C each prefer only one of the pieces, give A one of the pieces they don't want. Reassemble the two remaining pieces if needed to be re-divided between B and C using divide-and-choose.

- Case 2: Either B or C finds at least two pieces acceptable, say C. Then B, who only finds one piece acceptable, makes the first choice, C makes the second choice, and A takes the remaining piece.

Steinhaus lone-divider procedure for three

- Explain why the procedure produces a proportional allocation.
- It satisfies *none* of the other three criteria in general!

Steinhaus lone-divider procedure for three

Try it!

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