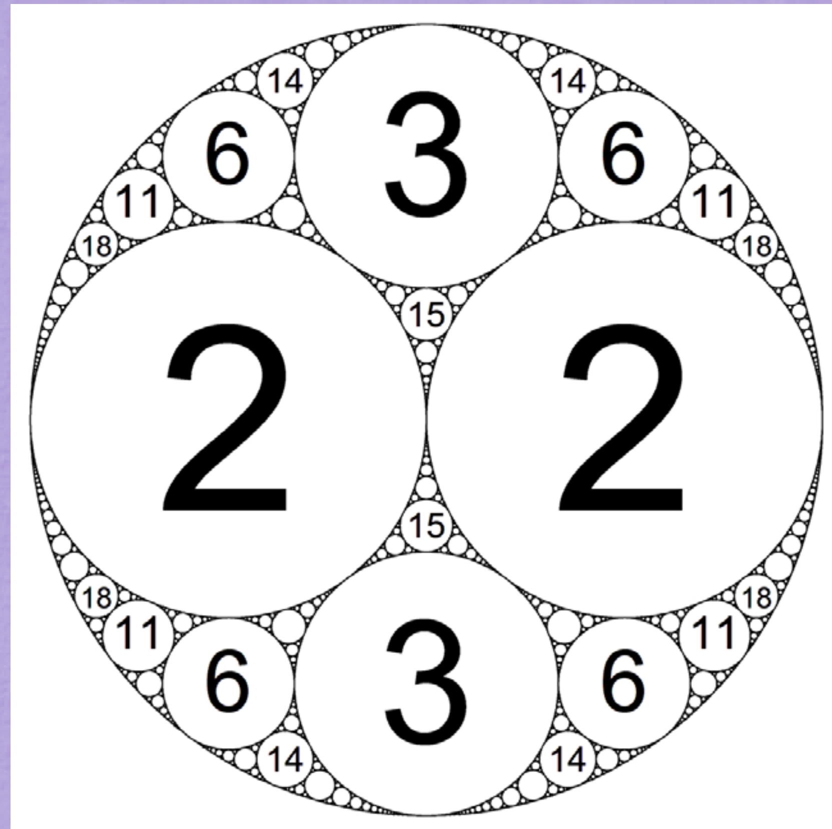


What's in a Logo?

Chris Goff – University of the Pacific

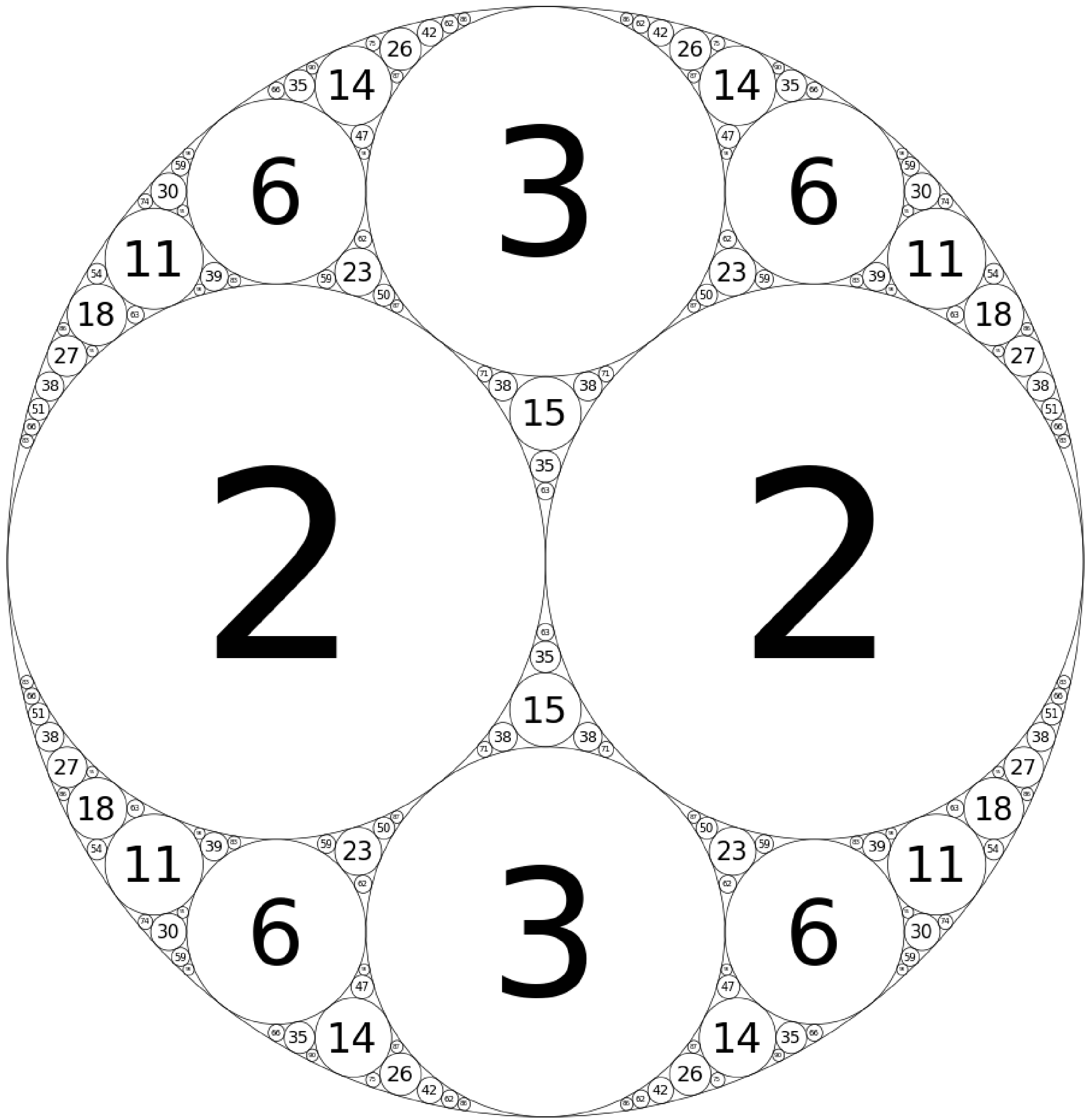
JMM Atlanta – January 6, 2017

Finding a logo

- Other MTC logos
- Other pictures involving circles
- Found Apollonian gaskets

Apollonian Gasket

- Region is filled by circles, possibly of different sizes
- A fractal
- Sage code, Java applet online
- Studied by number theorists



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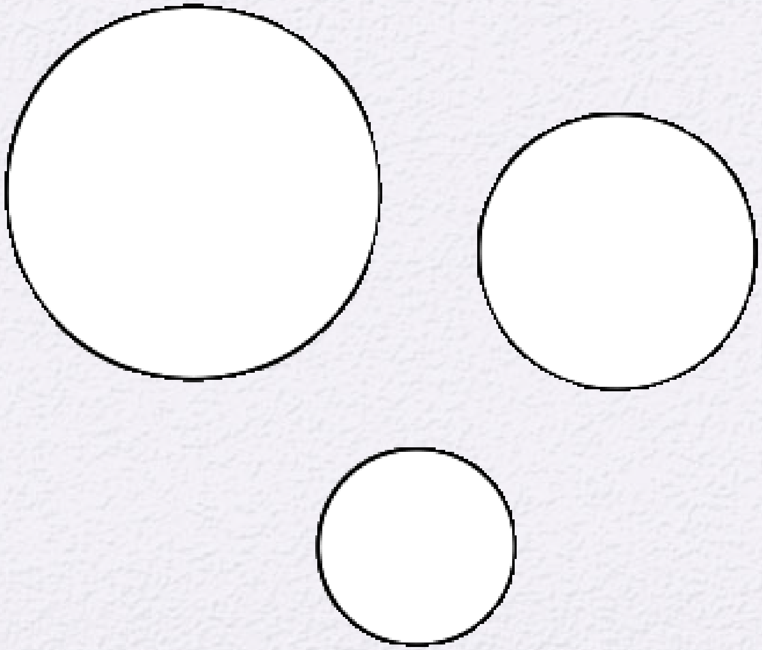
Questions I Had

- Why is this named for Apollonius?
  - The Circle Problem
- What do the numbers mean?
  - Curvature
  - Descartes formula
- What are some extensions of this construction?
  - Ford circles

Circle Problem

Find a circle that is tangent to all three given circles.

How many such circles can you find?



Apollonius of Perge (c.262-c.190 BC)

- Ancient Greek mathematician
- Born in southern Asia Minor
- Educated in Alexandria (?)
- Not really sure when he lived

Conics

Most famous extant work (8 books: first four are still in original Greek, next three were translated to Arabic in 9th C., 8th book lost)

- First definition of the double cone
- First to use terms "ellipse" and "hyperbola"
- First to treat ellipse, parabola, and hyperbola as sections of the same cone

Tangencies

- Source of the Circle Problem
- Lost (no versions have been found)

How did the Circle Problem survive?

- Pappus of Alexandria (c.290-c.350 AD)
  - Commented on Euclid, Apollonius, and Ptolemy
  - Also created his own theorems, extensions
  - Mentioned the Circle Problem of Apollonius
- Later solutions by van Roomen (1596), Viète, Descartes, Newton, Euler, Gauss, Cauchy, etc.

Princess Elisabeth of Bohemia (1618-1680)

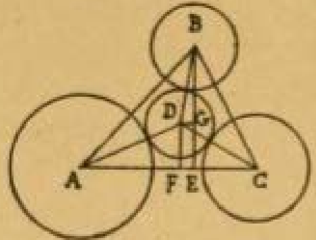
- Born in Heidelberg, grew up in the Netherlands, later ran an abbey
- Discussed philosophy with Descartes, as well as mathematics
- Asked Descartes how to find the radius of a circle that is tangent to three mutually tangent circles

René Descartes (1596-1650)

- French, lived in the Netherlands
- “I think, therefore I am”
- Cartesian coordinates
- Answered Princess Elisabeth’s question in 1643 (Vol. 4 of his complete works)

Excerpt 1, 1643

pour DF, l'autre pour DG, & l'autre pour le rayon du cercle cherché, i'ay tous les coftez des trois triangles rectangles ADF, BDG, CDF, | qui me donnent trois équations, pour ce qu'en chacun d'eux le quarré de la bafe eft égal aux deux quarrez des coftez.



Apres auoir ainfi fait autant d'équations que i'ay fuppofé de quantitez inconnuës, ie confidere fi, par chaque équation, i'en puis trouuer vne en termes affez fimples; & fi ie ne le puis, ie tafche d'en venir à bout, en ioignant deux ou plufieurs équations par l'addition ou fouftraction; & enfin, lors que cela ne fuffit pas, i'examine feulement s'il ne fera point mieux de changer les termes en quelque façon. Car, en faifant cét examen auec addreffe, on rencontre aifément les plus courts chemins, & on en peut effayer vne infinité en fort peu de temps.

Ainfi, en cét exemple, ie fuppofe que les trois bafes des triangles rectangles font[a]

$$\mathrm{AD} \infty a + x,$$
$$\mathrm{BD} \infty b + x,$$
$$\mathrm{CD} \infty c + x,$$

et, faifant $\mathrm{AE} \infty d$, $\mathrm{BE} \infty e$, $\mathrm{CE} \infty f$,

$$\mathrm{DF} \text{ ou } \mathrm{GE} \infty y, \quad \mathrm{DG} \text{ ou } \mathrm{FE} \infty z,$$

a. Clerselier emploie, comme signe d'égalité, les deux barres verticales,

$$\mathrm{AF} \infty d - z \ \& \ \mathrm{FD} \infty y,$$
$$\mathrm{BG} \infty e - y \ \& \ \mathrm{DG} \infty z,$$
$$\mathrm{CF} \infty f + z \ \& \ \mathrm{FD} \infty y.$$

Puis, faifant le quarré de chacune de ces bafes égal au quarré des deux coftez, i'ay les trois équations fuiuantes :

$$aa + 2ax + xx \infty dd - 2dz + zz + yy,$$
$$bb + 2bx + xx \infty ee - 2ey + yy + zz,$$
$$cc + 2cx + xx \infty ff + 2fz + zz + yy,$$

& ie voy que, par l'vne d'elles toute feule, ie ne puis trou|uer aucune des quantitez inconnuës, fans en tirer la racine quarrée, ce qui embarrafferoit trop la queftion. C'eft pourquoy ie viens au fecond moyen, qui eft de ioindre deux équations enfemble, & i'apperçois incontinent que, les termes xx, yy & zz eftant femblables en toutes trois, fi i'en ofte vne d'vne autre, laquelle ie voudray, ils s'effaceront, & ainfi ie n'auray plus de termes inconnus que x, y & z tous fimples. Ie voy auffi que, fi i'ofte la feconde de la premiere ou de la troifiéme, i'auray tous ces trois termes x, y & z; mais que, fi i'ofte la premiere de la troifiéme, ie n'auray que x & z. Ie choifis donc ce dernier chemin, & ie trouue

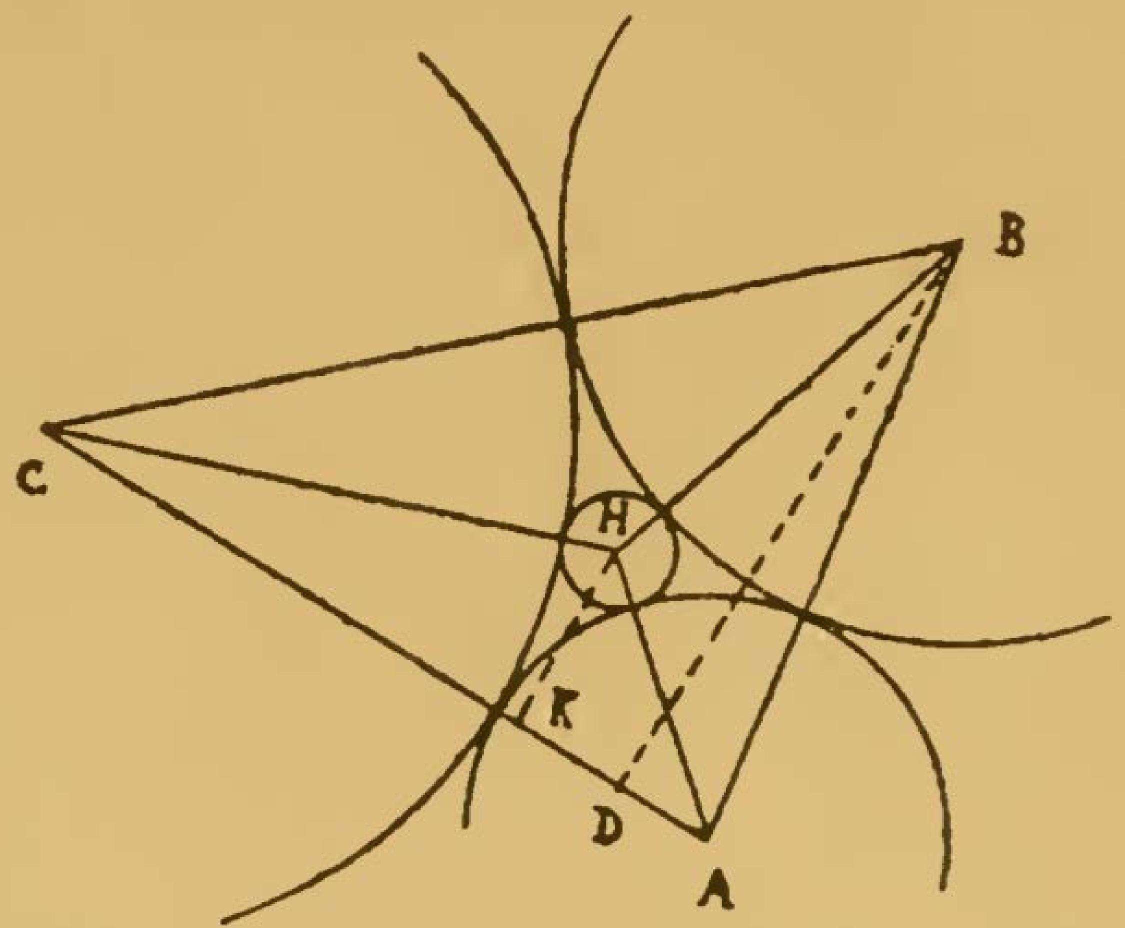
$$cc + 2cx - aa - 2ax \infty ff + 2fz - dd + 2dz,$$

ou bien $z \infty \frac{cc - aa + dd - ff + 2cx - 2ax}{2d + 2f}$,

ou bien $\frac{1}{2}d - \frac{1}{2}f + \frac{cc - aa + 2cx - 2ax}{2d + 2f}$.

Excerpt 2, 1643

Et en ſuiuant le calcul auec ces ſix lettres, ſans les changer ny en adjoûter d'autres, par le chemin qu'a



Curvature and Sign

- Definition: plus/minus the reciprocal of the radius (concave/convex)
- Concept can be extended to straight lines.
- Formula:

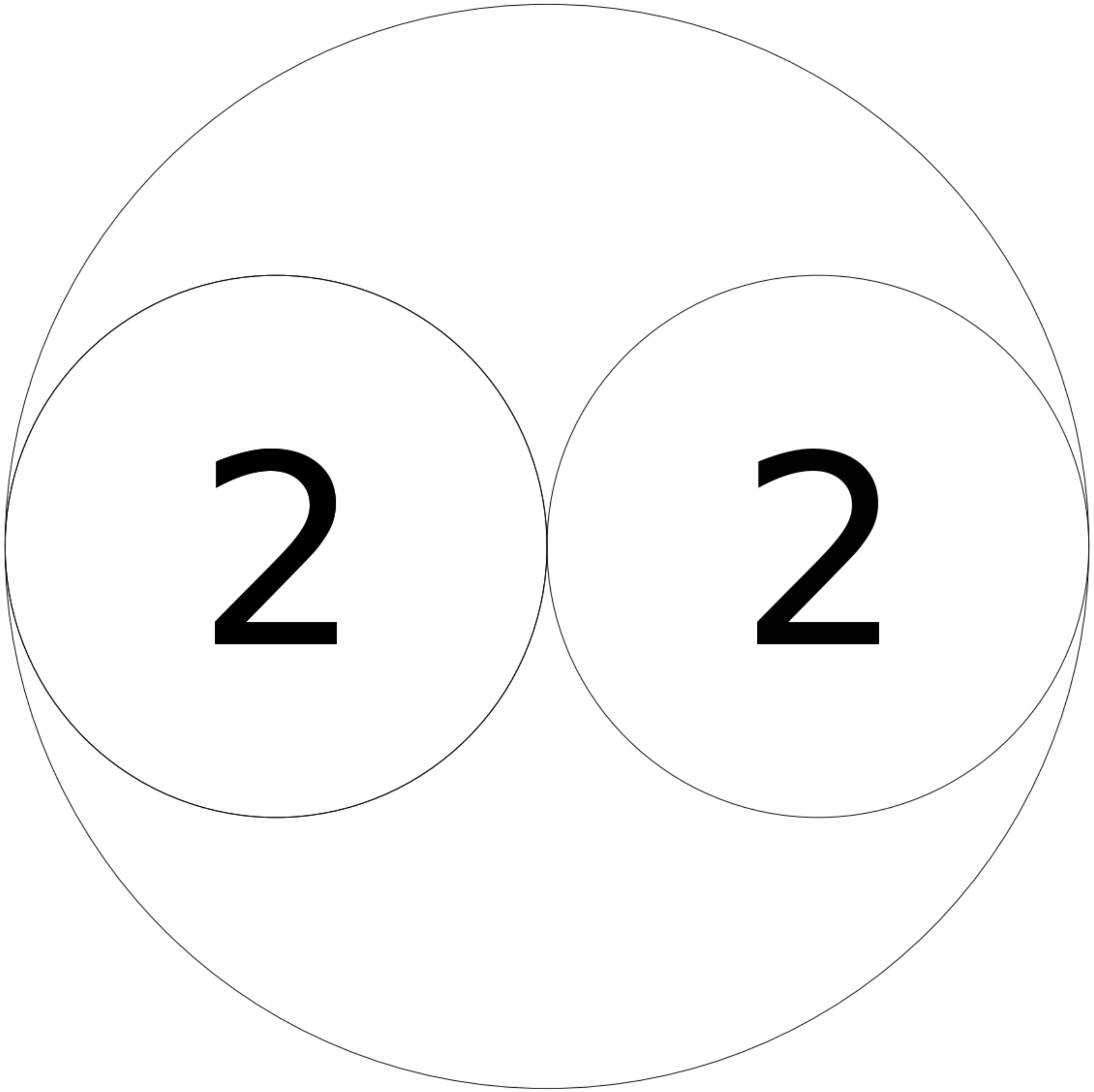
$$k = \pm \frac{1}{r}$$

Solution

- If there are four mutually tangent circles, then their curvatures satisfy:

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2)$$

- Even works if one circle is a straight line!


2
2

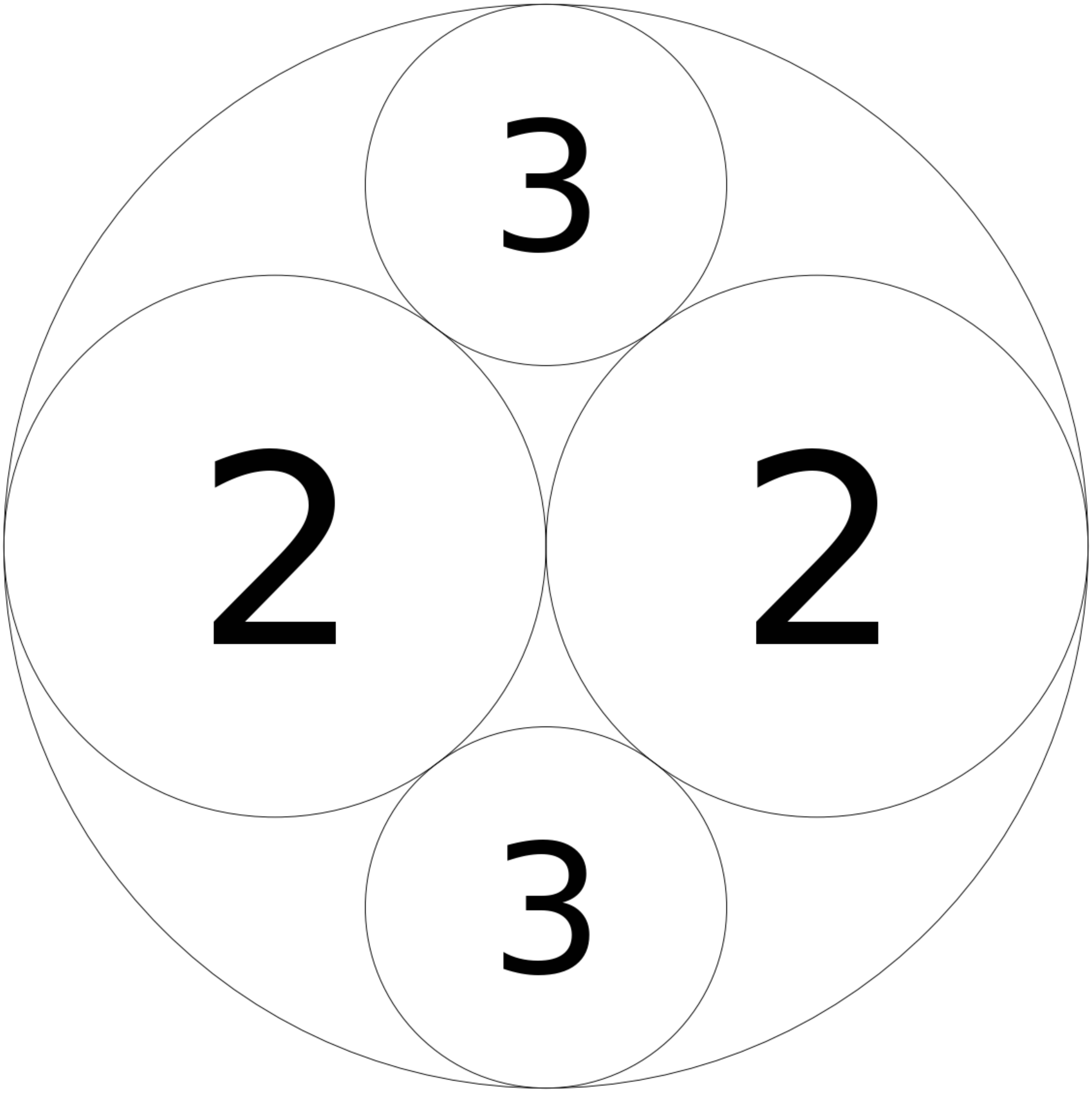
$$\left(k_1+k_2+k_3+k_4\right)^2 = 2(k_1^2+k_2^2+k_3^2+k_4^2)$$

- Note: the curvature of the big circle is –1.
- Given –1, 2, 2, what comes next?

$$\left(-1+2+2+k\right)^2 = 2(1+4+4+k^2)$$

$$9+6k+k^2 = 18+2k^2$$

$$0 = k^2-6k+9 = (k-3)^2$$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3

Explore

$$\left(k_1+k_2+k_3+k_4\right)^2=2(k_1^2+k_2^2+k_3^2+k_4^2)$$

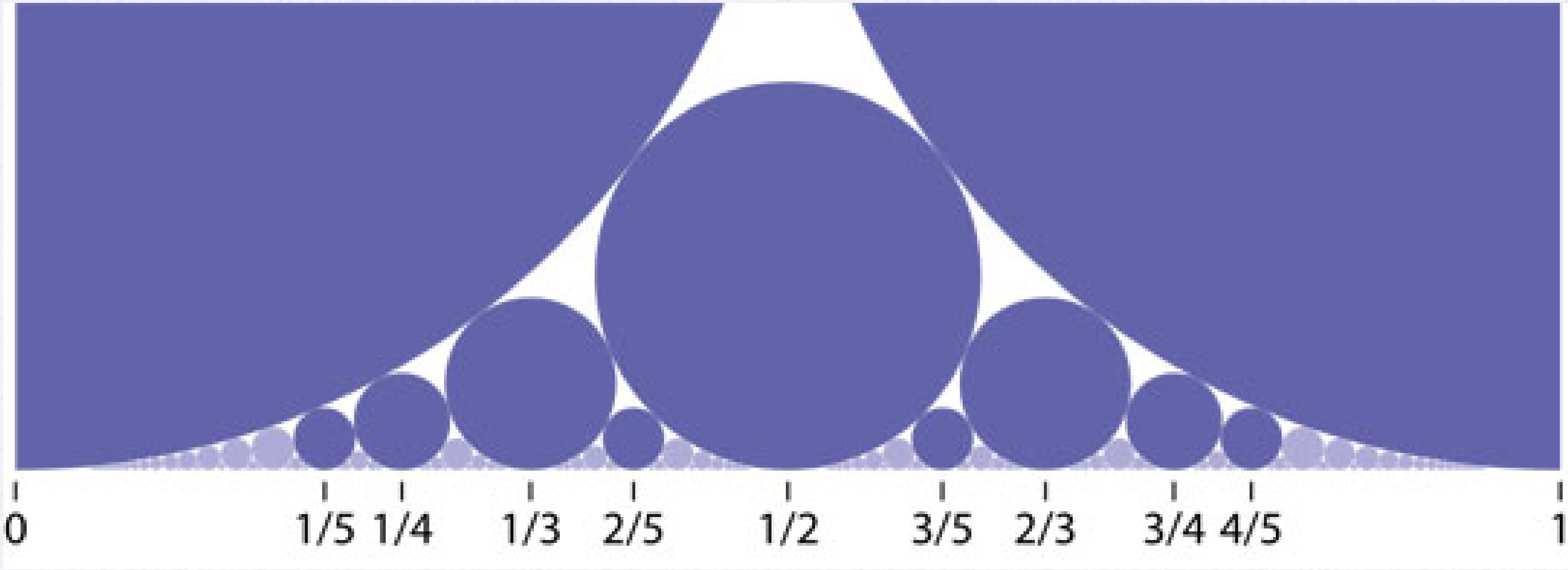
- Find more circles
- Start another using (-3, 5, 8, ?) or (-6, 10, 15, ?)
- Start your own

Extensions

- Why are we always getting integers for curvatures? Is that always true? If not, then what conditions make it true?
- Is there a similar formula for spheres?
- What if you have circles that sit on the integers on the number line?
- ???

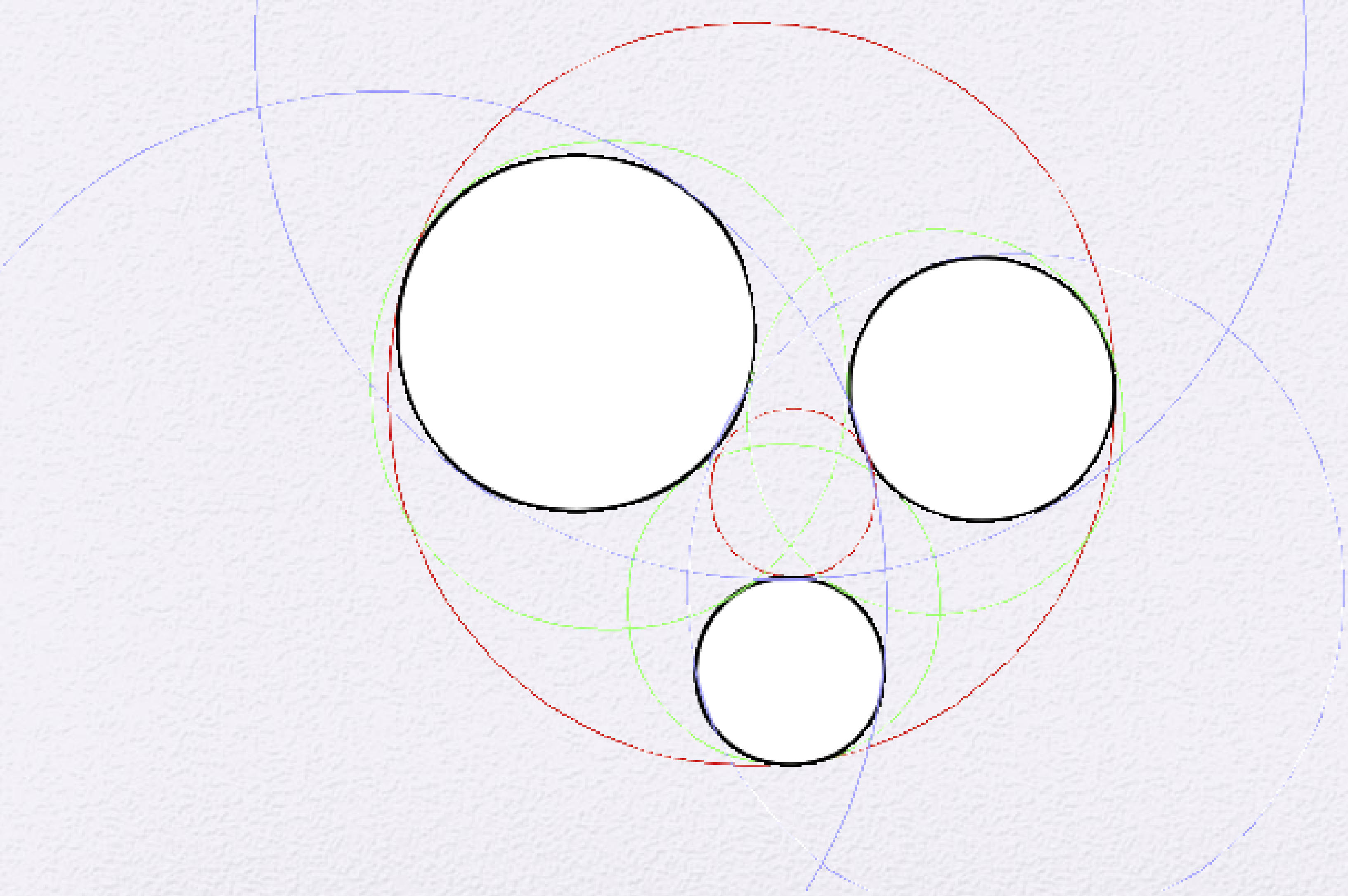
Ford Circles

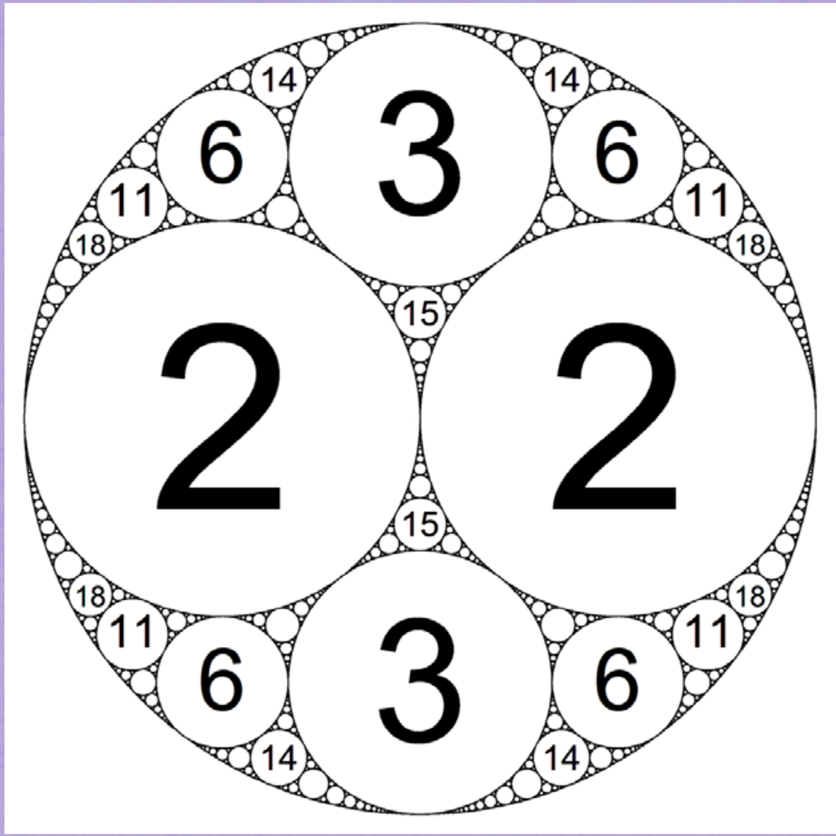
http://www.mathblog.dk/project-euler-73-sorted-reduced-proper-fractions/



Find a circle that is tangent to all three given circles.

How many such circles can you find?





What's in a Logo?

cgoff@pacific.edu

Thank you!