Math Wrangle Practice Problems II Solutions

American Mathematics Competitions

December 22, 2011

1. **Answer (839):** Note that

$$\frac{((3!)!)!}{3!} = \frac{(6!)!}{6} = \frac{720!}{6} = \frac{720 \cdot 719!}{6} = 120 \cdot 719!.$$

Because $120 \cdot 719! < 720!$, conclude that n must be less than 720, so the maximum value of n is 719. The requested value of $k+n$ is therefore $120 + 719 = 839$.

2. **Answer (301):** The sum of the areas of the green regions is

$$\begin{aligned} & \left[(2^2-1^2)+(4^2-3^2)+(6^2-5^2)+\cdots+(100^2-99^2)\right]\pi \\ = & \left[(2+1)+(4+3)+(6+5)+\cdots+(100+99)\right]\pi \\ = & \frac{1}{2}\cdot 100 \cdot 101\pi. \end{aligned}$$

Thus the desired ratio is

$$\frac{1}{2}\cdot\frac{100\cdot 101\pi}{100^2\pi} = \frac{101}{200},$$

and $m+n = 301$.

3. **Answer (484):** Since each element x of $\mathcal{S}$ is paired exactly once with every other element in the set, the number of times x contributes to the sum is the number of other elements in the set that are smaller than x. For example, the first number, 8, will contribute four times to the sum because the greater elements of the subsets $\{8,5\}$, $\{8,1\}$, $\{8,3\}$, and $\{8,2\}$ are all 8. Since the order of listing the elements in the set is not significant, it is helpful to first sort the elements of the set in increasing order. Thus, since $\mathcal{S} = \{1,2,3,5,8,13,21,34\}$, the sum of the numbers on the list is $0(1)+1(2)+2(3)+3(5)+4(8)+5(13)+6(21)+7(34) = 484$.

4. **Answer (012):** Use logarithm properties to obtain $\log(\sin x \cos x) = -1$, and then $\sin x \cos x = 1/10$. Note that

$$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x \cos x = 1 + \frac{2}{10} = \frac{12}{10}.$$

Thus

$$2\log(\sin x + \cos x) = \log\frac{12}{10} = \log 12 - 1,$$

so

$$\log(\sin x + \cos x) = \frac{1}{2}(\log 12 - 1),$$

and $n = 12$.

5. **Answer (505):** First consider the points in the six parallelepipeds projecting 1 unit outward from the original parallelepiped. Two of these six parallelepipeds are 1 by 3 by 4, two are 1 by 3 by 5, and two are 1 by 4 by 5. The sum of their volumes is $2(1\cdot 3\cdot 4 + 1\cdot 3\cdot 5 + 1\cdot 4\cdot 5) = 94$. Next consider the points in the twelve quarter-cylinders of radius 1 whose heights are the edges of the original parallelepiped. The sum of their volumes is $4\cdot\frac{1}{4}\pi\cdot 1^2(3+4+5) = 12\pi$. Finally, consider the points in the eight octants of a sphere of radius 1 at the eight vertices of the original parallelepiped. The sum of their volumes is $8\cdot\frac{1}{8}\cdot\frac{4}{3}\pi\cdot 1^3 = \frac{4\pi}{3}$. Because the volume of the original parallelepiped is $3\cdot 4\cdot 5 = 60$, the requested volume is $60 + 94 + 12\pi + 4\pi/3 = \dfrac{462 + 40\pi}{3}$, so $m + n + p = 462 + 40 + 3 = 505$.

6. **Answer (348):** The sides of the triangles may be cube edges, face-diagonals of length $\sqrt{2}$, or space-diagonals of length $\sqrt{3}$. A triangle can consist of two adjacent edges and a face-diagonal; three face-diagonals; or an edge, a face-diagonal, and a space-diagonal. The first type of triangle is right with area $1/2$, and there are 24 of them, 4 on each face. The second type of triangle is equilateral with area $\sqrt{3}/2$. There are 8 of these because each of these triangles is uniquely determined by the three vertices adjacent to one of the 8 vertices of the cube. The third type of triangle is right with area $\sqrt{2}/2$. There are 24 of these because there are four space-diagonals and each determines six triangles, one with each cube vertex that is not an endpoint of the diagonal. (Note that there is a total of $\binom{8}{3} = 56$ triangles with the desired vertices, which is consistent with the above results.) The desired sum is thus $24(1/2) + 8(\sqrt{3}/2) + 24(\sqrt{2}/2) = 12 + 4\sqrt{3} + 12\sqrt{2} = 12 + \sqrt{48} + \sqrt{288}$, and $m + n + p = 348$.

7. **Answer (380):** Let $AD = CD = a$, let $BD = b$, and let E be the projection of D on $\overline{AC}$. It follows that $a^2 - 15^2 = DE^2 = b^2 - 6^2$, or $a^2 - b^2 = 225 - 36 = 189$. Then $(a+b, a-b) = (189, 1)$, $(63, 3)$, $(27, 7)$, or $(21, 9)$, from which $(a, b) = (95, 94)$, $(33, 30)$, $(17, 10)$, or $(15, 6)$. The last pair is rejected since b must be greater than 6. Because each possible triangle has a perimeter of $2a + 30$, it follows that $s = 190 + 66 + 34 + 3\cdot 30 = 380$.

OR

Let (a_k, b_k) be the possible values for a and b, and let n be the number of possible perimeters of $\triangle ACD$. Then $s = \sum_{k=1}^{n}(30+2a_k) = 30n + \sum_{k=1}^{n}[(a_k+b_k)+(a_k-b_k)]$. But $(a_k+b_k)(a_k-b_k) = a_k^2 - b_k^2 = 189 = 3^3 \cdot 7$ which has 4 pairs of factors. Thus $n = 4$. Therefore the sum of the perimeters of the triangles is $30 \cdot 4$ more than the sum of the divisors of 189, that is, $120 + (1+3+3^2+3^3)(1+7) = 440$. However, this includes the case where $D = E$, the projection of D on $\overline{AC}$, so $s = 440 - 60 = 380$.

8. **Answer (129):** Let a, $a+d$, $a+2d$, and $\dfrac{(a+2d)^2}{a+d}$ be the terms of the sequence, with a and d positive integers. Then $(a+30)(a+d) = (a+2d)^2$, which yields $3a(10-d) = 2d(2d-15)$. It follows that either $10-d > 0$ and $2d-15 > 0$ or $10-d < 0$ and $2d-15 < 0$. In the first case, d is 8 or 9, and the second case has no solutions. When $d = 8$, $a = 8/3$, and when $d = 9$, $a = 18$. Thus, the only acceptable sequence is 18, 27, 36, 48, and the sum is 129.

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Answer (615): For a balanced four-digit integer, the sum of the leftmost two digits must be at least 1 and at most 18. Let $f(n)$ be the number of ways to write n as a sum of two digits where the first is at least 1, and let $g(n)$ be the number of ways to write n as a sum of two digits. For example, $f(3) = 3$, since $3 = 1+2 = 2+1 = 3+0$, and $g(3) = 4$. Then

$f(n) = \begin{cases} n & \text{for} 1 \le n \le 9, \\ 19-n & \text{for} 10 \le n \le 18, \end{cases}$ and $g(n) = \begin{cases} n+1 & \text{for} 1 \le n \le 9, \\ 19-n & \text{for} 10 \le n \le 18. \end{cases}$ For any balanced four-digit integer whose leftmost and rightmost digit pairs both have sum n, the number of possible leftmost digit pairs is $f(n)$ because the leftmost digit must be at least 1, and the number of possible rightmost digit pairs is $g(n)$. Thus there are $f(n) \cdot g(n)$ four-digit balanced integers whose leftmost and rightmost digit pairs both have sum n. The total number of balanced four-digit integers is then equal to

$$
\begin{aligned}
\sum_{n=1}^{18} f(n) \cdot g(n) &= \sum_{n=1}^{9} n(n+1) + \sum_{n=10}^{18}(19-n)^2 = \sum_{n=1}^{9}(n^2+n) + \sum_{n=1}^{9} n^2 \\
&= 2\sum_{n=1}^{9} n^2 + \sum_{n=1}^{9} n = 2(1^2+2^2+\cdots+9^2) + (1+2+\cdots+9) \\
&= 615.
\end{aligned}
$$

9. **Answer (083):** Let $\overline{CP}$ be the altitude to side $\overline{AB}$. Extend $\overline{AM}$ to meet $\overline{CP}$ at point L, as shown. Since $\angle ACL = 53^\circ$, conclude that $\angle MCL = 30^\circ$. Also,

$\angle LMC = \angle MAC + \angle ACM = 30°$. Thus $\triangle MLC$ is isosceles with $LM = LC$ and $\angle MLC = 120°$. Because L is on the perpendicular bisector of $\overline{AB}$, $\angle LBA = \angle LAB = 30°$ and $\angle MLB = 120°$. It follows that $\angle BLC = 120°$. Now consider $\triangle BLM$ and $\triangle BLC$. They share $\overline{BL}$, $ML = LC$, and $\angle MLB = \angle CLB = 120°$. Therefore they are congruent, and $\angle LMB = \angle LCB = 53°$. Hence $\angle CMB = \angle CML + \angle LMB = 30° + 53° = 83°$.

OR

Without loss of generality, assume that $AC = BC = 1$. Apply the Law of Sines in $\triangle AMC$ to obtain

$$\frac{\sin 150°}{1} = \frac{\sin 7°}{CM},$$

from which $CM = 2\sin 7°$. Apply the Law of Cosines in $\triangle BMC$ to obtain $MB^2 = 4\sin^2 7° + 1 - 2 \cdot 2\sin 7° \cdot \cos 83° = 4\sin^2 7° + 1 - 4\sin^2 7° = 1$. Thus $CB = MB$, and $\angle CMB = 83°$.

10. **Answer (336):** Call the three integers a, b, and c, and, without loss of generality, assume $a \le b \le c$. Then $abc = 6(a+b+c)$, and $c = a+b$. Thus $abc = 12c$, and $ab = 12$, so $(a,b,c) = (1,12,13)$, $(2,6,8)$, or $(3,4,7)$, and $N = 156$, 96, or 84. The sum of the possible values of N is 336.

11. **Answer (120):** An integer is divisible by 8 if and only if the number formed by the rightmost three digits is divisible by 8. The greatest integer with the desired property is formed by choosing 9876543 as the seven leftmost digits and finding the arrangement of 012 that yields the greatest multiple of 8, assuming that such an arrangement exists. Checking the 6 permutations of 012 yields 120 as the sole multiple of 8, so $N = 9876543120$, and its remainder when divided by 1000 is 120.

12. **Answer (192):** There are three choices for the first letter and two choices for each subsequent letter, so there are $3 \cdot 2^{n-1}$ n-letter good words. Substitute $n = 7$ to find there are $3 \cdot 2^6 = 192$ seven-letter good words.

13. **Answer (028):** Let O and P be the centers of faces DAB and ABC, respectively, of regular tetrahedron $ABCD$. Both $\overrightarrow{DO}$ and $\overrightarrow{CP}$ intersect $\overline{AB}$ at its midpoint M. Since $\dfrac{MO}{MD} = \dfrac{MP}{MC} = \dfrac{1}{3}$, triangles MOP and MDC are similar, and $OP = (1/3)DC$.

Because the tetrahedra are similar, the ratio of their volumes is the cube of the ratio of a pair of corresponding sides, namely, $(1/3)^3 = 1/27$, so $m + n = 28$.

14. **Answer (216):** Let $\overline{AB}$ be a diameter of the circular face of the wedge formed by the first cut, and let $\overline{AC}$ be the longest chord across the elliptical face of the wedge formed by the second cut. Then $\triangle ABC$ is an isosceles right triangle and $BC = 12$ inches. If a third cut were made through the point C on the log and perpendicular to the axis of the cylinder, then a second wedge, congruent to the original, would be formed, and the two wedges would fit together to form a right circular cylinder with radius $AB/2 = 6$ inches and height BC. Thus, the volume of the wedge is $\frac{1}{2}\pi \cdot 6^2 \cdot 12 = 216\pi$, and $n = 216$.

15. **Answer (112):** Let M be the midpoint of $\overline{BC}$, let M' be the reflection of M in G, and let Q and R be the points where $\overline{BC}$ meets $\overline{A'C'}$ and $\overline{A'B'}$, respectively. Note that since M is on $\overline{BC}$, M' is on $\overline{B'C'}$. Because a 180° rotation maps each line that does not contain the center of the rotation to a parallel line, $\overline{BC}$ is parallel to $\overline{B'C'}$, and $\triangle A'RQ$ is similar to $\triangle A'B'C'$. Recall that medians of a triangle trisect each other to obtain

$$M'G = MG = (1/3)AM, \text{ so } A'M = AM' = (1/3)AM = (1/3)A'M'.$$

Thus the similarity ratio between triangles $A'RQ$ and $A'B'C'$ is $1/3$, and

$$[A'RQ] = (1/9)[A'B'C'] = (1/9)[ABC].$$

Similarly, the area of each of the two small triangles with vertices at B' and C', respectively, is $1/9$ that of $\triangle ABC$. The desired area is therefore

$$[ABC] + 3(1/9)[ABC] = (4/3)[ABC].$$

Use Heron's formula, $K = \sqrt{s(s-a)(s-b)(s-c)}$, to find $[ABC] = \sqrt{21 \cdot 7 \cdot 6 \cdot 8} = 84$. The desired area is then $(4/3) \cdot 84 = 112$.

16. **Answer (400):** Let O be the point of intersection of diagonals $\overline{AC}$ and $\overline{BD}$, and E the point of intersection of $\overline{AC}$ and the circumcircle of $\triangle ABD$. Extend $\overline{DB}$ to meet the circumcircle of $\triangle ACD$ at F. From the Power-of-a-Point Theorem, we have

$$AO \cdot OE = BO \cdot OD \text{ and } DO \cdot OF = AO \cdot OC.$$

Let $AC = 2m$ and $BD = 2n$. Because $\overline{AE}$ is a diameter of the circumcircle of $\triangle ABD$, and $\overline{DF}$ is a diameter of the circumcircle of $\triangle ACD$, the above equalities can be rewritten as

$$m(25 - m) = n^2 \quad \text{and} \quad n(50 - n) = m^2,$$

or

$$25m = m^2 + n^2 \quad \text{and} \quad 50n = m^2 + n^2.$$

Therefore $m = 2n$. It follows that $50n = 5n^2$, so $n = 10$ and $m = 20$. Thus $[ABCD] = (1/2)AC \cdot BD = 2mn = 400$.

OR

Let R_1 and R_2 be the circumradii of triangles ABD and ACD, respectively. Because $\overline{BO}$ is the altitude to the hypotenuse of right $\triangle ABE$, $AB^2 = AO \cdot AE$. Similarly, in right $\triangle DAF$, $AB^2 = DA^2 = DO \cdot DF$, so $AO \cdot AE = DO \cdot DF$. Thus

$$\frac{AO}{DO} = \frac{DF}{AE} = \frac{R_2}{R_1} = 2.$$

Also, from right $\triangle ADE$, $2 = \dfrac{AO}{DO} = \dfrac{DO}{OE}$. Then

$$25 = 2R_1 = AE = AO + OE = 2 \cdot DO + \frac{1}{2}DO = \frac{5}{2}DO,$$

and $DO = 10$, $AO = 20$, so $[ABCD] = 400$.

OR

Let s be the length of a side of the rhombus, and let $\alpha = \angle BAC$. Then $AO = s\cos\alpha$, and $BO = s\sin\alpha$, so $[ABCD] = 4[ABO] = 2s^2 \sin\alpha\cos\alpha = s^2\sin 2\alpha$. Apply the Extended Law of Sines (In any $\triangle ABC$ with $AB = c$, $BC = a$, $CA = b$, and circumradius R, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$) in $\triangle ABD$ and $\triangle ACD$ to obtain $s = 2R_1\sin(90^\circ - \alpha) = 2R_1\cos\alpha$, and $s = 2R_2\sin\alpha$. Thus $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{R_1}{R_2} = \dfrac{1}{2}$. Also, $s^2 = 4R_1R_2\cos\alpha\sin\alpha = 2R_1R_2\sin 2\alpha$. But $\sin 2\alpha = 2 \cdot \dfrac{1}{\sqrt{5}} \cdot \dfrac{2}{\sqrt{5}} = \dfrac{4}{5}$, from which $[ABCD] = 2R_1R_2\sin^2 2\alpha = 2 \cdot \frac{25}{2} \cdot 25 \cdot \frac{16}{25} = 400$.

OR

Let $AB = s$, $AO = m$, and $BO = n$, and use the fact that the product of the lengths of the sides of a triangle is four times the product of its area and its circumradius to obtain $4[ABD]R_1 = s \cdot s \cdot 2n$ and $4[ACD]R_2 = s \cdot s \cdot 2m$. Since $[ABD] = [ACD]$, conclude that $\dfrac{1}{2} = \dfrac{R_1}{R_2} = \dfrac{n}{m}$, and proceed as above.

17. **Answer (348):** The nth term of an arithmetic sequence has the form $a_n = pn + q$, so the product of corresponding terms of two arithmetic sequences is a quadratic expression, $s_n = an^2 + bn + c$. Letting $n = 0$, 1, and 2 produces the equations $c = 1440$, $a+b+c = 1716$, and $4a+2b+c = 1848$, whose common solution is $a = -72$, $b = 348$, and $c = 1440$. Thus the eighth term is $s_7 = -72 \cdot 7^2 + 348 \cdot 7 + 1440 = 348$. Note that $s_n = -72n^2 + 348n + 1440 = -12(2n - 15)(3n + 8)$ can be used to generate pairs of arithmetic sequences with the desired products, such as $\{180, 156, 132, \ldots\}$ and $\{8, 11, 14, \ldots\}$.

18. **Answer (006):** Apply the division algorithm for polynomials to obtain

$$P(x) = Q(x)(x^2 + 1) + x^2 - x + 1.$$

Therefore

$$\sum_{i=1}^{4} P(z_i) = \sum_{i=1}^{4} z_i^2 - \sum_{i=1}^{4} z_i + 4 = \Big(\sum_{i=1}^{4} z_i\Big)^2 - 2\sum_{i<j} z_i z_j - \sum_{i=1}^{4} z_i + 4.$$

Use the formulas for sum and product of the roots to obtain $\sum_{i=1}^{4} P(z_i) = 1+2-1+4 = 6$.

OR

Since, for each root w of $Q(x) = 0$, we have $w^4 - w^3 - w^2 - 1 = 0$, conclude that $w^4 - w^3 = w^2 + 1$, and then $w^6 - w^5 = w^4 + w^2 = w^3 + 2w^2 + 1$. Thus $P(w) = w^3 + 2w^2 + 1 - w^3 - w^2 - w = w^2 - w + 1$. Therefore

$$\sum_{i=1}^{4} P(z_i) = \sum_{i=1}^{4} z_i^2 - \sum_{i=1}^{4} z_i + 4,$$

and, as above, $\sum_{i=1}^{4} P(z_i) = 6$.

19. **Answer (156):** Let x represent the smaller of the two integers. Then $\sqrt{x} + \sqrt{x+60} = \sqrt{y}$, and $x + x + 60 + 2\sqrt{x(x+60)} = y$. Thus $x(x+60) = z^2$ for some positive integer z. It follows that

$$x^2 + 60x = z^2,$$
$$x^2 + 60x + 900 = z^2 + 900,$$

$$(x+30)^2 - z^2 = 900, \quad \text{and}$$
$$(x+30+z)(x+30-z) = 900.$$

Thus $(x+30+z)$ and $(x+30-z)$ are factors of 900 with $(x+30+z) > (x+30-z)$, and they are both even because their sum and product are even. Note that each pair of even factors of 900 can be found by doubling factor-pairs of 225, so the possible values of $(x+30+z, x+30-z)$ are $(450, 2)$, $(150, 6)$, $(90, 10)$, and $(50, 18)$. Each of these pairs yields a value for x which is 30 less than half their sum. These values are 196, 48, 20, and 4. When $x = 196$ or 4, then $\sqrt{x} + \sqrt{x+60}$ is an integer. When $x = 48$, we obtain $\sqrt{48} + \sqrt{108} = \sqrt{300}$, and when $x = 20$, we obtain $\sqrt{20} + \sqrt{80} = \sqrt{180}$. Thus the desired maximum sum is $48 + 108 = 156$.

OR

Let x represent the smaller of the two integers. Then $\sqrt{x} + \sqrt{x+60} = \sqrt{y}$, and $x + x + 60 + 2\sqrt{x(x+60)} = y$. Thus $x(x+60) = z^2$ for some positive integer z. Let d be the greatest common divisor of x and $x+60$. Then $x = dm$ and $x + 60 = dn$, where m and n are relatively prime. Because $dm \cdot dn = z^2$, there are relatively prime positive integers p and q such that $m = p^2$ and $n = q^2$. Now $d(q^2 - p^2) = 60$. Note that p and q cannot both be odd, else $q^2 - p^2$ would be divisible by 8; and they cannot both be even because they are relatively prime. Therefore p and q are of opposite parity, and $q^2 - p^2$ is odd, which implies that $q^2 - p^2 = 1, 3, 5$, or 15. But $q^2 - p^2$ cannot be 1, and if $q^2 - p^2$ were 15, then d would be 4, and x and $x + 60$ would be squares. Thus $q^2 - p^2 = 3$ or 5, and $(q+p, q-p) = (3, 1)$ or $(5, 1)$, and then $(q, p) = (2, 1)$ or $(3, 2)$. This yields $(x+60, x) = (2^2 \cdot 20, 1^2 \cdot 20) = (80, 20)$ or $(x+60, x) = (3^2 \cdot 12, 2^2 \cdot 12) = (108, 48)$, so the requested maximum sum is $108 + 48 = 156$.

OR

Let a and b represent the two integers, with $a > b$. Then $a - b = 60$, and $\sqrt{a} + \sqrt{b} = \sqrt{c}$, where c is an integer that is not a square. Dividing yields $\sqrt{a} - \sqrt{b} = 60/\sqrt{c}$. Adding these last two equations yields

$$2\sqrt{a} = \sqrt{c} + \frac{60}{\sqrt{c}}, \quad \text{so}$$
$$2\sqrt{ac} = c + 60.$$

Therefore $\sqrt{ac}$ is an integer, so c is even, as is ac, which implies $\sqrt{ac}$ is even. Hence c is a multiple of 4, so there is a positive non-square integer d such that $c = 4d$. Then

$$a = \frac{(c+60)^2}{4c} = \frac{(4d+60)^2}{16d} = \frac{(d+15)^2}{d} = \frac{d^2 + 30d + 225}{d} = d + \frac{225}{d} + 30.$$

Thus d is a non-square divisor of 225, so the possible values of d are 3, 5, 15, 45, and 75. The maximum value of a, which occurs when $d = 3$ or $d = 75$, is $3+75+30 = 108$, so the maximum value of b is $108 - 60 = 48$, and the requested maximum sum is $48 + 108 = 156$.

21. **Answer (217):** Let the digits of n, read from left to right, be a, $a-1$, $a-2$, and $a-3$, respectively, where a is an integer between 3 and 9, inclusive. Then $n = 1000a + 100(a-1) + 10(a-2) + a - 3 = 1111a - 123 = 37(30a-4) + (a+25)$, where $0 \le a + 25 < 37$. Thus the requested sum is

$$\sum_{a=3}^{9}(a+25) = \left(\sum_{a=3}^{9} a\right) + 175 = 42 + 175 = 217.$$

OR

There are seven such four-digit integers, the smallest of which is 3210, whose remainder when divided by 37 is 28. The seven integers form an arithmetic sequence with common difference 1111, whose remainder when divided by 37 is 1, so the sum of the remainders is $28 + 29 + 30 + 31 + 32 + 33 + 34 = 7 \cdot 31 = 217$.

22. **Answer (201):** Let the smallest elements of $\mathcal{A}$ and $\mathcal{B}$ be $(n+1)$ and $(k+1)$, respectively. Then

$$2m = (n+1) + (n+2) + \cdots + (n+m) = mn + \frac{1}{2} \cdot m(m+1), \quad \text{and}$$
$$m = (k+1) + (k+2) + \cdots + (k+2m) = 2km + \frac{1}{2} \cdot 2m(2m+1).$$

The second equation implies that $k+m = 0$. Substitute this into $|k+2m-(n+m)| = 99$ to obtain $n = \pm 99$. Now simplify the first equation to obtain $2 = n + (m+1)/2$, and substitute $n = \pm 99$. This yields $m = -195$ or $m = 201$. Because $m > 0$, $m = 201$.

OR

The mean of the elements in $\mathcal{A}$ is 2, and the mean of the elements in $\mathcal{B}$ is $1/2$. Because the mean of each of these sets equals its median, and the median of $\mathcal{A}$ is an integer, m is odd. Thus $\mathcal{A} = \{2 - \frac{m-1}{2}, \ldots, 2, \ldots, 2 + \frac{m-1}{2}\}$, and $\mathcal{B} = \{-m+1, \ldots, 0, 1, \ldots, m\}$. Therefore $|2 + \frac{m-1}{2} - m| = 99$, which yields $|\frac{3-m}{2}| = 99$, so $|3 - m| = 198$. Because $m > 0$, $m = 201$.

23. **Answer (241):** The total number of diagonals and edges is $\binom{26}{2} = 325$, and there are $12 \cdot 2 = 24$ face diagonals, so P has $325 - 60 - 24 = 241$ space diagonals. One such

polyhedron can be obtained by gluing two dodecahedral pyramids onto the 12-sided faces of a dodecahedral prism.

Note that one can determine that there are 60 edges as follows. The 24 triangles contribute $3 \cdot 24 = 72$ edges, and the 12 quadrilaterals contribute $4 \cdot 12 = 48$ edges. Because each edge is in two faces, there are $\frac{1}{2}(72 + 48) = 60$ edges.

24. **Answer (086):** Let $\overline{PQ}$ be a line segment in set $\mathcal{S}$ that is not a side of the square, and let M be the midpoint of $\overline{PQ}$. Let A be the vertex of the square that is on both the side that contains P and the side that contains Q. Because $\overline{AM}$ is the median to the hypotenuse of right $\triangle PAQ$, $AM = (1/2) \cdot PQ = (1/2) \cdot 2 = 1$. Thus every midpoint is 1 unit from a vertex of the square, and the set of all the midpoints forms four quarter-circles of radius 1 and with centers at the vertices of the square. The area of the region bounded by the four arcs is $4 - 4 \cdot (\pi/4) = 4 - \pi$, so $100k = 100(4 - 3.14) = 86$.

OR

Place a coordinate system so that the vertices of the square are at $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$. When the segment's vertices are on the sides that contain $(0,0)$, its endpoints' coordinates can be represented as $(a,0)$ and $(0,b)$. Let the coordinates of the midpoint of the segment be (x,y). Then $(x,y) = (a/2, b/2)$ and $a^2 + b^2 = 4$. Thus $x^2 + y^2 = (a/2)^2 + (b/2)^2 = 1$, and the midpoints of these segments form a quarter-circle with radius 1 centered at the origin. The set of all the midpoints forms four quarter-circles, and the area of the region bounded by the four arcs is $4 - 4 \cdot (\pi/4) = 4 - \pi$, so $100k = 100(4 - 3.14) = 86$.

25. **Answer (849):** Let Beta's scores be a out of b on day one and c out of d on day two, so that $0 < a/b < 8/15$, $0 < c/d < 7/10$, and $b + d = 500$. Then $(15/8)a < b$ and $(10/7)c < d$, so $(15/8)a + (10/7)c < b + d = 500$, and $21a + 16c < 5600$. Beta's two-day success ratio is greatest when $a + c$ is greatest. Let $M = a + c$ and subtract $16M$ from both sides of the last inequality to obtain $5a < 5600 - 16M$. Because $a > 0$, conclude that $5600 - 16M > 0$, and $M < 350$. When $M = 349$, $5a < 16$, so $a \leq 3$. If $a = 3$, then $b \geq 6$, but then $d \leq 494$ and $c = 346$ so $c/d \geq 346/494 > 7/10$. Notice that when $a = 2$ and $b = 4$, then $a/b < 8/15$ and $c/d = 347/496 < 7/10$. Thus Beta's maximum possible two-day success ratio is $349/500$, so $m + n = 849$.

OR

Let M be the total number of points scored by Beta in the two days. Notice first that $M < 350$, because 350 is 70% of 500, and Beta's success ratio is less than 70% on each day of the competition. Notice next that $M = 349$ is possible, because Beta could score 1 point out of 2 attempted on the first day, and 348 out of 498 attempted on the second day. Thus $m = 349$, $n = 500$, and $m + n = 849$.

Note that Beta's two-day success ratio can be greater than Alpha's while Beta's success ratio is less on each day. This is an example of Simpson's Paradox.

26. **Answer (882):** To find the number of snakelike numbers that have four different digits, distinguish two cases, depending on whether or not 0 is among the chosen digits. For the case where 0 is not among the chosen digits, first consider only the digits 1, 2, 3, and 4. There are exactly 5 snakelike numbers with these digits: 1423, 1324, 2314, 2413, and 3412. There are $\binom{9}{4} = 126$ ways to choose four non-zero digits and five ways to arrange each such set for a total of 630 numbers. In the other case, there are $\binom{9}{3} = 84$ ways to choose three digits to go with 0, and three ways to arrange each set of four digits, because the snakelike numbers with the digits 0, 1, 2, and 3 would correspond to the list above, but with the first two entries deleted. There are $84 \cdot 3 = 252$ such numbers. Thus there are $630 + 252 = 882$ four-digit snakelike numbers with distinct digits.

27. **Answer (588):** Each of the x^2-terms in the expansion of the product is obtained by multiplying the x-terms from two of the 15 factors of the product. The coefficient of the x^2-term is therefore the sum of the products of each pair of numbers in the set $\{-1, 2, -3, \ldots, 14, -15\}$. Note that, in general,

$$(a_1 + a_2 + \cdots + a_n)^2 = a_1^2 + a_2^2 + \cdots + a_n^2 + 2 \cdot \left(\sum_{1 \le i < j \le n} a_i a_j \right).$$

Thus

$$\begin{aligned} C = \sum_{1 \le i < j \le 15} (-1)^i i (-1)^j j &= \frac{1}{2} \left(\left(\sum_{k=1}^{15} (-1)^k k \right)^2 - \sum_{k=1}^{15} k^2 \right) \\ &= \frac{1}{2} \left((-8)^2 - \frac{15(15+1)(2 \cdot 15 + 1)}{6} \right) = -588. \end{aligned}$$

Hence $|C| = 588$.

OR

Note that

$$f(x) = (1 - x)(1 + 2x)(1 - 3x) \ldots (1 - 15x)$$

$$\begin{aligned} &= 1 + (-1 + 2 - 3 + \cdots - 15)x + Cx^2 + \cdots \\ &= 1 - 8x + Cx^2 + \cdots . \end{aligned}$$

Thus $f(-x) = 1 + 8x + Cx^2 - \cdots$.
But $f(-x) = (1 + x)(1 - 2x)(1 + 3x) \ldots (1 + 15x)$, so

$$\begin{aligned} f(x)f(-x) &= (1 - x^2)(1 - 4x^2)(1 - 9x^2) \ldots (1 - 225x^2) \\ &= 1 - (1^2 + 2^2 + 3^2 + \cdots + 15^2)x^2 + \cdots . \end{aligned}$$

Also $f(x)f(-x) = (1 - 8x + Cx^2 + \cdots)(1 + 8x + Cx^2 - \cdots) = 1 + (2C - 64)x^2 + \cdots$.
Thus $2C - 64 = -(1^2 + 2^2 + 3^3 + \cdots + 15^2)$, and, as above, $|C| = 588$.

28. **Answer (199):** Let $\mathcal{C}$ be the circle determined by P_1, P_2, and P_3. Because the path turns counterclockwise at an angle of less than 180° at P_2 and P_3, P_1 and P_4 must be on the same side of line P_2P_3. Note that $\triangle P_1P_2P_3 \cong \triangle P_4P_3P_2$, and so $\angle P_2P_1P_3 \cong \angle P_3P_4P_2$. Thus P_4 is on $\mathcal{C}$. Similarly, because P_2, P_3, and P_4 are on $\mathcal{C}$, P_5 must be too, and, in general, $P_1, P_2, \ldots, P_n$ are all on $\mathcal{C}$. The fact that the minor arcs $P_1P_2, P_2P_3, \ldots,$ and P_nP_1 are congruent implies that $P_1, P_2, \ldots,$ and P_n are equally spaced on $\mathcal{C}$.

Thus any regular n-pointed star can be constructed by choosing n equally spaced points on a circle, and numbering them consecutively from 0 to $n - 1$. For positive integers $d < n$, the path consisting of line segments whose vertices are numbered $0, d, 2d, \ldots, (n-1)d, 0$ modulo n will be a regular n-pointed star if and only if $2 \leq d \leq n - 2$ and d is relatively prime to n. This is because if $d = 1$ or $d = n - 1$, the resulting path will be a polygon; and if d is not relatively prime to n, not every vertex will be included in the path. Also, for any choice of d that yields a regular n-pointed star, any two such stars will be similar because a dilation of one of the stars about the center of its circle will yield the other.

Because $1000 = 2^3 \cdot 5^3$, numbers that are relatively prime to 1000 are those that are multiples of neither 2 nor 5. There are $1000/2 = 500$ multiples of 2 that are less than or equal to 1000; there are $1000/5 = 200$ multiples of 5 that are less than or equal to 1000; and there are $1000/10 = 100$ numbers less than or equal to 1000 that are multiples of both 2 and 5. Hence there are $1000 - (500 + 200 - 100) = 400$ numbers that are less than 1000 and relatively prime to 1000, and 398 of them are between 2 and 998, inclusive. Because $d = k$ yields the same path as $d = n - k$ (and also because one of these two paths turns clockwise at each vertex), there are $398/2 = 199$ non-similar regular 1000-pointed stars.

9. **Answer (035):** Let s_1 be the line segment drawn in $\triangle ABC$, and let s_2 be the line segment drawn in rectangle $DEFG$. To obtain a triangle and a trapezoid, line

segment s_2 must pass through exactly one vertex of rectangle $DEFG$. Hence V_2 is a trapezoid with a right angle, and U_2 is a right triangle. Therefore line segment s_1 is parallel to one of the legs of $\triangle ABC$ and, for all placements of s_1, U_1 is similar to $\triangle ABC$. It follows that there are two possibilities for triangle U_2: one in which the sides are 6, 9/2, and 15/2; and the other in which the sides are 7, 21/4, and 35/4. Were s_1 parallel to the side of length 4, trapezoids V_1 and V_2 could not be similar, because the corresponding acute angles in V_1 and V_2 would not be congruent; but when s_1 is parallel to the side of length 3, the angles of trapezoid V_1 are congruent to the corresponding angles of V_2, so it is possible to place segment s_1 so that V_1 is similar to V_2. In the case when the triangle U_2 has sides 6, 9/2, and 15/2, the bases of trapezoid V_2 are 7 and $7-(9/2)=5/2$, so its bases, and therefore the bases of V_1, are in the ratio $5:14$. Then the area of triangle U_1 is $(5/14)^2 \cdot (1/2) \cdot 3 \cdot 4 = 75/98$. In the case when the triangle U_2 has sides 7, 21/4, and 35/4, the bases of trapezoid V_2 are 6 and $6-(21/4)=3/4$, so its bases, and the bases of V_1, are in the ratio $1:8$. The area of triangle U_1 is then $(1/8)^2 \cdot (1/2) \cdot 3 \cdot 4 = 3/32$. The minimum value of the area of U_1 is thus 3/32, and $m+n=35$.

30. **Answer (817):** In order for the circle to lie completely within the rectangle, the center of the circle must lie in a rectangle that is $(15-2)$ by $(36-2)$ or 13 by 34. The requested probability is equal to the probability that the distance from the circle's center to the diagonal $\overline{AC}$ is greater than 1, which equals the probability that the distance from a randomly selected point in the 13-by-34 rectangle to each of the sides of $\triangle ABC$ and $\triangle CDA$ is greater than 1. Let $AB=36$ and $BC=15$. Draw the three line segments that are one unit respectively from each of the sides of $\triangle ABC$ and whose endpoints are on the sides. Let E, F, and G be the three points of intersection nearest to A, B, and C, respectively, of the three line segments. Let P be the intersection of $\overrightarrow{EG}$ and $\overline{BC}$, and let G' and P' be the projections of G and P on $\overline{BC}$ and $\overline{AC}$, respectively. Then $FG = BC - CP - PG' - 1$. Notice that $\triangle PP'C \sim \triangle ABC$ and $PP'=1$, so $CP = AC/AB$. Similarly, $\triangle GG'P \sim \triangle ABC$ and $GG'=1$, so $PG' = CB/AB$. Thus

$$FG = BC - \frac{AC}{AB} - \frac{CB}{AB} - 1.$$

Apply the Pythagorean Theorem to $\triangle ABC$ to obtain $AC=39$. Substitute these lengths to find that $FG=25/2$. Notice that $\triangle EFG \sim \triangle ABC$, and their similarity ratio is $(25/2)/15 = 5/6$, so $[EFG] = (25/36)[ABC]$. The requested probability is therefore

$$\frac{2 \cdot \frac{25}{36} \cdot \frac{1}{2} \cdot 15 \cdot 36}{13 \cdot 34} = \frac{375}{442},$$

so $m+n=817$.

OR

Define E, F, and G as in the previous solution. Each of these three points is equidistant from two sides of $\triangle ABC$, and they are therefore on the angle-bisectors of angles A, B, and C, respectively. These angle-bisectors are also angle-bisectors of $\triangle EFG$ because its sides are parallel to those of $\triangle ABC$. Thus $\triangle ABC$ and $\triangle EFG$ have the same incenter, and the inradius of $\triangle EFG$ is one less than that of $\triangle ABC$. In general, the inradius of a triangle is the area divided by one-half the perimeter, so the inradius of $\triangle ABC$ is 6. The similarity ratio of $\triangle EFG$ to $\triangle ABC$ is the same as the ratio of their inradii, namely $5/6$. Continue as in the previous solution.

OR

Define E, F, G, and G' as in the previous solutions. Notice that $\overline{CG}$ bisects $\angle ACB$ and that $\cos\angle ACB = 5/13$, and so, by the Half-Angle Formula, $\cos\angle GCG' = 3/\sqrt{13}$. Thus, for some x, $CG' = 3x$ and $CG = x\sqrt{13}$. Apply the Pythagorean Theorem in $\triangle CG'G$ to conclude that $(x\sqrt{13})^2 - (3x)^2 = 1$, so $x = 1/2$. Then $CG' = 3x = 3/2$, and $FG = 15 - 1 - 3/2 = 25/2$. Continue as in the first solution.

31. **Answer (592):** Without loss of generality, let the radius of the circle be 2. The radii to the endpoints of the chord, along with the chord, form an isosceles triangle with vertex angle 120°. The area of the larger of the two regions is thus 2/3 that of the circle plus the area of the isosceles triangle, and the area of the smaller of the two regions is thus 1/3 that of the circle minus the area of the isosceles triangle. The requested ratio is therefore $\dfrac{\frac{2}{3}\cdot 4\pi + \sqrt{3}}{\frac{1}{3}\cdot 4\pi - \sqrt{3}} = \dfrac{8\pi + 3\sqrt{3}}{4\pi - 3\sqrt{3}}$, so $abcdef = 8\cdot 3\cdot 3\cdot 4\cdot 3\cdot 3 = 2592$, and the requested remainder is 592.

32. **Answer (441):** In order for Terry and Mary to get the same color combination, they must select all red candies or all blue candies, or they must each select one of each color. The probability of getting all red candies is $\dfrac{\binom{10}{2}\binom{8}{2}}{\binom{20}{2}\binom{18}{2}} = \dfrac{10\cdot 9\cdot 8\cdot 7}{20\cdot 19\cdot 18\cdot 17}$. The probability of getting all blue candies is the same. The probability that they each select one of each color is $\dfrac{10^2\cdot 9^2}{\binom{20}{2}\binom{18}{2}} = \dfrac{10^2\cdot 9^2\cdot 4}{20\cdot 19\cdot 18\cdot 17}$. Thus the probability of getting the same combination is

$$2\cdot\frac{10\cdot 9\cdot 8\cdot 7}{20\cdot 19\cdot 18\cdot 17} + \frac{10^2\cdot 9^2\cdot 4}{20\cdot 19\cdot 18\cdot 17} = \frac{10\cdot 9\cdot 8\cdot(14+45)}{20\cdot 19\cdot 18\cdot 17} = \frac{2\cdot 59}{19\cdot 17} = \frac{118}{323},$$

and $m + n = 441$.

33. **Answer (384):** Let the dimensions of the block be p cm by q cm by r cm. The invisible cubes form a rectangular solid whose dimensions are $p-1$, $q-1$, and $r-1$. Thus $(p-1)(q-1)(r-1) = 231$. There are only five ways to write 231 as a product of three positive integers:

$$231 = 3 \cdot 7 \cdot 11 = 1 \cdot 3 \cdot 77 = 1 \cdot 7 \cdot 33 = 1 \cdot 11 \cdot 21 = 1 \cdot 1 \cdot 231$$

The corresponding blocks are $4 \times 8 \times 12$, $2 \times 4 \times 78$, $2 \times 8 \times 34$, $2 \times 12 \times 22$, and $2 \times 2 \times 232$. Their volumes are 384, 624, 544, 528, and 928, respectively. Thus the smallest possible value of N is 384.

34. **Answer (927):** There are 99 numbers with the desired property that are less than 100. Three-digit numbers with the property must have decimal representations of the form aaa, aab, aba, or abb, where a and b are digits with $a \geq 1$ and $a \neq b$. There are 9 of the first type and $9 \cdot 9 = 81$ of each of the other three. Four-digit numbers with the property must have decimal representations of the form $aaaa$, $aaab$, $aaba$, $aabb$, $abaa$, $abab$, $abba$, or $abbb$. There are 9 of the first type and 81 of each of the other seven. Thus there are a total of $99 + 9 + 3 \cdot 81 + 9 + 7 \cdot 81 = 927$ numbers with the desired property.

OR

Count the number of positive integers less than 10,000 that contain at least 3 distinct digits. There are $9 \cdot 9 \cdot 8$ such 3-digit integers. The number of 4-digit integers that contain exactly 3 distinct digits is $\binom{4}{2} \cdot 9 \cdot 9 \cdot 8$ because there are $\binom{4}{2}$ choices for the positions of the two digits that are the same, 9 choices for the digit that appears in the first place, and 9 and 8 choices for the two other digits, respectively. The number of 4-digit integers that contain exactly 4 distinct digits is $9 \cdot 9 \cdot 8 \cdot 7$. Thus there are $9 \cdot 9 \cdot 8 + 6 \cdot 9 \cdot 9 \cdot 8 + 9 \cdot 9 \cdot 8 \cdot 7 = 14 \cdot 9 \cdot 9 \cdot 8 = 9072$ positive integers less than 10,000 that contain at least 3 distinct digits, and there are $9999 - 9072 = 927$ integers with the desired property.

35. **Answer (766):** Choose a unit of time so that the job is scheduled to be completed in 4 of these units. The first quarter was completed in 1 time unit. For the second quarter of the work, there were only 9/10 as many workers as in the first quarter, so it was completed in 10/9 units. For the third quarter, there were only 8/10 as many workers as in the first quarter, so it was completed in 5/4 units. This leaves $4 - (1 + 10/9 + 5/4) = 23/36$ units to complete the final quarter. To finish the job on schedule, the number of workers that are needed is at least 36/23 of the number of workers needed in the first quarter, or $(36/23)1000$ which is between 1565 and 1566. There are 800 workers at the end of the third quarter, so a minimum of $1566 - 800 = 766$ additional workers must be hired.

36. **Answer (408):** Let the first monkey take $8x$ bananas from the pile, keeping $6x$ and giving x to each of the others. Let the second monkey take $8y$ bananas from the pile, keeping $2y$ and giving $3y$ to each of the others. Let the third monkey take $24z$ bananas from the pile, keeping $2z$ and giving $11z$ to each of the others. The total number of bananas is $8x+8y+24z$. The given ratios imply that $6x+3y+11z=3(x+3y+2z)$ and $x+2y+11z=2(x+3y+2z)$. Simplify these equations to obtain $3x+5z=6y$ and $7z=x+4y$. Eliminate x to obtain $9y=13z$. Then $y=13n$ and $z=9n$, where n is a positive integer. Substitute to find that $x=11n$. Thus, the least possible values for x, y and z are 11, 13 and 9, respectively, and the least possible total is $8\cdot 11+8\cdot 13+24\cdot 9=408$.

37. **Answer (293):** Let $\overline{B'C'}$ and $\overline{CD}$ intersect at H. Note that $B'E=BE=17$. Apply the Pythagorean Theorem to $\triangle EAB'$ to obtain $AB'=15$. Because $\angle C'$ and $\angle C'B'E$ are right angles, $\triangle B'AE\sim\triangle HDB'\sim\triangle HC'F$, so the lengths of the sides of each triangle are in the ratio $8:15:17$. Now $C'F=CF=3$ implies that $FH=(17/8)3=51/8$ and $DH=25-(3+51/8)=125/8$. Then $B'D=(8/15)(125/8)=25/3$. Thus $AD=70/3$, and the perimeter of $ABCD$ is

$$2\cdot 25+2\cdot\frac{70}{3}=\frac{290}{3},$$

so $m+n=290+3=293$.

OR

Notice first that $B'E=BE=17$. Apply the Pythagorean Theorem to $\triangle EAB'$ to obtain $AB'=15$. Draw $\overline{FG}$ parallel to $\overline{CB}$, with G on $\overline{AB}$. Notice that $GE=17-3=14$. Because points on the crease $\overline{EF}$ are equidistant from B and B', it follows that $\overline{EF}$ is perpendicular to $\overline{BB'}$, and hence that triangles EGF and $B'AB$ are similar. In particular, $\dfrac{FG}{BA}=\dfrac{GE}{AB'}$. This yields $FG=70/3$, and the perimeter of $ABCD$ is therefore $290/3$.

38. **Answer (054):** A positive integer N is a divisor of 2004^{2004} if and only if $N=2^i3^j167^k$ with $0\le i\le 4008$, $0\le j\le 2004$, and $0\le k\le 2004$. Such a number has exactly 2004 positive integer divisors if and only if $(i+1)(j+1)(k+1)=2004$. Thus the number of values of N meeting the required conditions is equal to the number of ordered triples of positive integers whose product is 2004. Each of the unordered triples $\{1002,2,1\}$, $\{668,3,1\}$, $\{501,4,1\}$, $\{334,6,1\}$, $\{334,3,2\}$, $\{167,12,1\}$, $\{167,6,2\}$, and $\{167,4,3\}$ can be ordered in 6 possible ways, and the triples $\{2004,1,1\}$ and $\{501,2,2\}$ can each be ordered in 3 possible ways, so the total is $8\cdot 6+2\cdot 3=54$.

OR

Begin as above. Then, to find the number of ordered triples of positive integers whose product is 2004, represent the triples as $(2^{a_1} \cdot 3^{b_1} \cdot 167^{c_1}, 2^{a_2} \cdot 3^{b_2} \cdot 167^{c_2}, 2^{a_3} \cdot 3^{b_3} \cdot 167^{c_3})$, where $a_1 + a_2 + a_3 = 2$, $b_1 + b_2 + b_3 = 1$, and $c_1 + c_2 + c_3 = 1$, and the a_i's, b_i's, and c_i's are nonnegative integers. The number of solutions of $a_1 + a_2 + a_3 = 2$ is $\binom{4}{2}$ because each solution corresponds to an arrangement of two objects and two dividers. Similarly, the number of solutions of both $b_1 + b_2 + b_3 = 1$ and $c_1 + c_2 + c_3 = 1$ is $\binom{3}{1}$, so the total number of triples is $\binom{4}{2}\binom{3}{1}\binom{3}{1} = 6 \cdot 3 \cdot 3 = 54$.

39. **Answer (973):** The terms in the sequence are $1, r, r^2, r(2r-1), (2r-1)^2, (2r-1)(3r-2), (3r-2)^2, \ldots$. Assuming that the pattern continues, the ninth term is $(4r-3)^2$ and the tenth term is $(4r-3)(5r-4)$. Thus $(4r-3)^2 + (4r-3)(5r-4) = 646$. This leads to $(36r+125)(r-5) = 0$. Because the terms are positive, $r = 5$. Substitute to find that $a_n = (2n-1)^2$ when n is odd, and that $a_n = (2n-3)(2n+1)$ when n is even. The least odd-numbered term greater than 1000 is therefore $a_{17} = 33^2 = 1089$, and $a_{16} = 29 \cdot 33 = 957 < 1000$. The desired value of $n + a_n$ is $957 + 16 = 973$.

The pattern referred to above is

$$\begin{aligned} a_{2n} &= [(n-1)r - (n-2)][nr - (n-1)], \\ a_{2n+1} &= [nr - (n-1)]^2. \end{aligned}$$

This pattern has been verified for the first few positive integral values of n. The above equations imply that

$$\begin{aligned} a_{2n+2} &= 2[nr-(n-1)]^2 - [(n-1)r-(n-2)][nr-(n-1)] \\ &= [nr-(n-1)][2nr - 2(n-1) - (n-1)r + (n-2)] \\ &= [nr-(n-1)][(n+1)r - n], \quad \text{and} \\ a_{2n+3} &= \frac{[nr-(n-1)]^2[(n+1)r-n]^2}{[nr-(n-1)]^2} \\ &= [(n+1)r-n]^2. \end{aligned}$$

The above argument, along with the fact that the pattern holds for $n = 1$ and $n = 2$, implies that it holds for all positive integers n.

40. **Answer (913):** An element of S has the form $2^a + 2^b$, where $0 \le a \le 39$, $0 \le b \le 39$, and $a \ne b$, so S has $\binom{40}{2} = 780$ elements. Without loss of generality, assume $a < b$. Note that 9 divides $2^a + 2^b = 2^a(2^{b-a}+1)$ if and only if 9 divides $2^{b-a}+1$, that is, when $2^{b-a} \equiv 8 \pmod 9$. Because $2^1, 2^2, 2^3, 2^4, 2^5, 2^6$, and $2^7 \equiv 2, 4, 8, 7, 5, 1$, and 2 (mod 9), respectively, conclude that $2^{b-a} \equiv 8 \pmod 9$ when $b - a = 6k - 3$ for positive integers k. But $b - a = 6k - 3$ implies that $0 \le a \le 39 - (6k-3)$, so there are $40 - (6k-3) = 43 - 6k$ ordered pairs (a, b) that satisfy $b - a = 6k - 3$. Because $6k - 3 \le 39$, conclude that $1 \le k \le 7$. The number of multiples of 9 in S is therefore