The Difference Game

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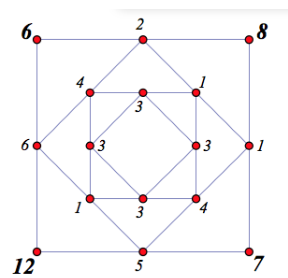
Start by labeling the vertices of a square with numbers. Then write the difference of the values at two adjacent vertices on the midpoint of the line joining them; this produces four new values at the vertices of a smaller square. Keep repeating this process, generating smaller and smaller squares until the process ends. In the example below, we started with the values $6, 8, 7, 12$ (shown in larger font) which generated the values $2, 1, 5, 6$, then $4, 1, 4, 1$. The final square shown has all vertices equal to 3; clearly the next square (and all subsequent squares) will have only zeros at each vertex.



Investigate, generate questions, come up with conjectures.

Solution: There are two big questions (at least). The first is whether you always end up with zeros eventually, and the second is whether you can hold out for an arbitrary length of time.

The answers are YES to both questions (provided you start with positive integers). To see why you eventually get only zeros, we will verify two simple observations.

- **The maximum of the four values NEVER goes up; it either stays the same or decreases.** This is pretty obvious. However, this is not enough to force the values to all become zero, since perhaps there could be some "oscillation." We need the next observation.

- **Eventually all the values will be even.** This is not obvious, but there are only 6 different parity cases to try, using 0 for even and 1 for odd: 0000,1000,1100,1010,1110,1111. With each case, we end up with all evens (all 0s) in at most 4 turns.

Thus, no matter what numbers you get, eventually, you will end up with a square whose values are all even. Call the values $2a, 2b, 2c, 2d$. Now, when you continue the process, everything is multiplied by this factor of 2, and you can visualize it as two identical squares playing the difference game. At some point, both of *those* squares will have all even values. But it was really just one square, so putting the factor of 2 back, we now have all values being *multiples of 4*.

This process will continue indefinitely, which forces the values to eventually be zeros, since no positive integer can have an arbitrarily high power of two dividing it!

Now that we know we will eventually get all zeros, we need to find a way to hold off that fate as long as possible. We can design a starting square that will take at least N turns to zero out, for any N. The secret is "tribonacci numbers," the sequence $1, 1, 1, 3, 5, 9, 17, \ldots$, where the first three terms are 1 and each subsequent term is the sum of the *three* preceding terms.

To see how this works, imagine that we put the values $t_{13}, t_{12}, t_{11}, t_{10}$ on the vertices of a square (clockwise, in that order), where t_n is the $n$the tribonacci number. After one turn, the vertices of the new square are

$$t_{13} - t_{12}, \quad t_{12} - t_{11}, \quad t_{11} - t_{10}, \quad t_{13} - t_{10}.$$

Using the definition of tribonacci numbers, we see that

$$t_{13} - t_{12} = (t_{12} + t_{11} + t_{10}) - t_{12} = t_{11} + t_{10}.$$

Likewise, $t_{12} - t_{11} = t_{10} + t_9$ and $t_{11} - t_{10} = t_9 + t_8$. The last difference is a slightly different pattern (since the indices differ by 3 instead of 1), but the magic of tribonacci yields

$$t_{13} - t_{10} = (t_{12} + t_{11} + t_{10}) - t_{10} = t_{12} + t_{11}.$$

It doesn't matter which vertex is "first," as long as we go in order. So observe that the values of this new square, going clockwise, are

$$t_{12} + t_{11}, \quad t_{11} + t_{10}, \quad t_{10} + t_9, \quad t_9 + t_8.$$

Now if we use the "two squares" idea, we see that this new square can really be thought of as one square with vertices of $t_{12}, t_{11}, t_{10}, t_9$, and the other, with corresponding vertices of t_{11}, t_{10}, t_9, t_8. Well, we know what happens when we take differences; these are just squares with consecutive tribonacci values (only shifted backwards a bit).

You can see the pattern. If we let S_n denote the square whose values are the consecutive tribonacci numbers $t_n, t_{n-1}, t_{n-2}, t_{n-3}$, we see that after one turn, S_n becomes "$S_{n-1} + S_{n-2}$," and after k turns, we will have a "sum" of 2^k tribonacci squares. The values of these tribonacci squares stay non-zero and non-constant until you hit S_4; after four turns that zeros out.

Notice that this "multiple squares" idea requires that the values of the squares be monotonic in the same direction; this allows us to couple or uncouple squares without interference. So at the very least, we can be assured of getting at least $n - 4$ turns if we start with S_n.