# Brahmagupta f1y1: *Bulgarian Solitaire   09.26.15, 09.29.15*

| Supplies | Notes |
|---|---|
| To copy:<br>**Indicate # of copies (1 per student? 1 per group?)**<br>● Blank tables   2 students per page (i.e. cut the pages in half)<br>Supplies:<br>● 10 * # of students of playing cards--we should have at least six decks of cards at each site | ● Author: Doug O'Roark (doug@mathcirclesofchicago.org)<br><br>Resources:<br>● 30 Years of Bulgarian Solitaire  Brian Hopkins<br>● The Last Recreations, "Bulgarian Solitaire and Other Seemingly Endless Tasks," Martin Gardner |

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|---|---|
| **Transferable Heuristics** | ● Make an organized table.<br>● Look for patterns. |
| **Objectives:** | ● Students will investigate the game Bulgarian solitaire and practice creating their own experiments.<br>● Students will compare different games and look for patterns and make conjectures. |
| **Teacher Overview** | Bulgarian Solitaire is a simple card game that is easy to learn but creates great opportunities for students to conjecture, explore, and justify ideas (with explanations that can be formulated by kids at this age group).<br>The rules of the game are simple. Arrange cards in piles, and in each 'turn' take a card from each pile and use those removed cards to make a new pile. For example, if we had piles of 7, 5, 2, after a turn we would have piles of 6, 4, 1, and 3 (we now have a 3 because there were 3 piles in the previous step).<br>Investigations are typically organized around a particular number of cards. The activity starts with 10 cards. No matter where you start, you will always reach the fixed point 4-3-2-1. Upon further investigation you'll see that any set of cards whose size is a triangular number will lead to a similar fixed point. Other numbers lead to cycles. |

Narrative

| Time/Description | Activity (include links to any handouts) | Teacher Notes |
|---|---|---|
| 1:00-1:10 | Stand, Make a group of 4, no one you've met before; memorize names. Teacher will call on one person at random from group, they have to say all names. You have 60 seconds--go!<br>Kids share<br>Teacher introduces themselves--get personal! Use the chance to show that you are friendly, not too serious, maybe know something 7th/8th graders would be interested in…. | You can ask kids to stand and find people they don't know, and stand while doing the activity. This allows for movement. Then, when you finish, they sit down, which gives a clear transition to the next activity where they sit in pairs, not groups. |
| Bulgarian solitaire intro<br>1:10-1:35 | <u>Everybody 10 cards; I'm going to teach you Bulgarian Solitaire</u>: [5]<br>● Make 2 piles, 7-3<br>● Take one card from each pile; make a new pile.<br>● Somebody tell me what you have.<br>● That's it' do it again, and again, and again.<br>Kids continue [2]<br>What do we notice? [3]<br><br><u>Ok, let's make a slight change. We played Bulgarian Solitaire, 10 cards, 7-3 split. What else could we do?</u><br>Students suggest.<br>Pull out 5-5, 6-4, 8-2, 9-1, 10-0 papers (note that kids could potentially suggest something like 4-3-3]<br>Students work in pairs, making sure they get the same result as their partner[10]<br><br><u>At some point interrupt groups: [3]</u><br>We need a representation. How could you represent this WITHOUT cards?<br>Be sure to insist on *descending order*<br>Then back to work. Be prepared to share your work on board. | It can be nice to point out that the choice of descending order does not change our results. What's the point? Easier communication (much like an insistence that polynomials be written in descending order.) |

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<u>Discussion [10]</u>
<ul>
<li>See if everyone is following:
<ul>
<li>3-3-2-2 What's next?</li>
<li>7--1-1-1 What's next?</li>
</ul>
</li>
<li>Have a couple of groups share their work on the board.</li>
<li>What's a way to check for errors? [Add the numbers; you should still get 10]</li>
<li>Somebody make a conjecture (looking for 4-3-2-1 is invariant; name this as a fixed point)</li>
<li>Did you ever see a pattern in one game that came up in another game (besides 4321)? How can you use that to save time?</li>
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In this part of the discussion, there is a great chance to make connections between the different partitions. That's what the last question here is about.<br><br>
Depending on time, one possible avenue for exploration is representing the stages in a Bulgarian Solitaire game as nodes on a digraph. By doing so, you can try and prove, for example, that any partition of 10 will lead to the fixed point 4-3-2-1.<br><br>
Later in the fall, students will look at Crossing the River with Dogs--type-problems, where they will also represent a problem with a digraph.
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<td>1:35-2:00<br>Other numbers of cards.</td>
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Ask students: How can we vary the game?<br>
Look for: vary number of cards.<br>
Ask kids to conjecture: Do you think you'll always get a fixed point for any number of cards? After you experiment, refine your conjecture!<br>
Groups of 4; each group explores two numbers. Assume 5 groups<br>
Group 1: 8 and 3<br>
Group 2: 4 and 6<br>
Group 3: 1 and 9<br>
Group 4: 7 and 2<br>
Group 5: 5 and 6<br>
If you finish your numbers, try others. [At this point you might choose to hand out blank tables for students to complete]<br>
By 1:50 have different kids complete a table you make on the board (they complete the 2nd and third columns of what's below. 10 would be
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<tr><td>n</td><td>fixed?</td><td>Pattern</td></tr>
<tr><td>1</td><td>yes</td><td>1</td></tr>
<tr><td>2</td><td>no</td><td>1-1, 2, 1-1, 2 (2 cycle)</td></tr>
<tr><td>3</td><td>yes</td><td>2-1</td></tr>
<tr><td>4</td><td>no</td><td>3 cycle</td></tr>
<tr><td>5</td><td>no</td><td>3 cycle</td></tr>
<tr><td>6</td><td>yes</td><td>3-2-1</td></tr>
<tr><td>7</td><td>no</td><td>4 cycle</td></tr>
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<td>completed by you.)<br><br>Wrap up:<ul><li>What patterns do you see in the table?</li><li>What's the next fixed point after 10? After that?</li><li>What helped us see the pattern? [An organized table, with the inputs in order, starting at 1, going up by 1/linearly]</li></ul></td>
<td><table><tr><td>8</td><td>no</td><td>4 cycle (or 2 cycle!)</td></tr><tr><td>9</td><td>no</td><td>4 cycle</td></tr><tr><td>10</td><td>yes</td><td>4-3-2-1</td></tr></table><br>Note: Kids are often confused by experimenting with 1, 2, or 3 cards, so they may need support to feel confident they are doing it correctly.</td>
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Suggested Extension Questions

- For kids that complete the investigation (and can predict which numbers lead to fixed points and the period of the cycles of other numbers), have them investigate 8 cards carefully. They'll find there are two possible cycle lengths. Are there any other such numbers?
- We might call an 'Adam' point a set that has no ancestors (above, 6,4,1,3 is not an Adam point). For a given number, what are the corresponding Adam points?
- What's the next fixed point after 1,3,6 and 10?
- Name a fixed point between…..
- Can you give a formula for the nth fix point, or a descriptive rule?
- Is there a way to predict how long a cycle will be for a given large number that doesn't lead to a fixed point?
- How long will the cycle for 43 cards be?
- Have students make a digraph for a particular n. For example, for 4 cards, the digraph would be equivalent to something like:
    - (1,1,1,1) → (4) → (3,1) → (2,2) → (points back at (3,1))
    - (2,1,1) → (like 4, points to (3,1))
    - Note that (1,1,1,1) is the only Adam Point for n = 4.
    - The digraph proves by exhaustion that no matter where you start you end up in the (3,1) → (2,2) cycle
- Investigate 8 carefully. Can you find any other numbers like 8?