Math Circle Demonstration: Fun with Grids and Open-ended Investigations

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The following problems are examples of some low-threshold high-ceiling questions that were successfully investigated in our Math Circle. Students who are curious about patterns will naturally ask follow-up questions, and start working on generalizations and extensions. In Math Circles we let the students explore and satisfy their curiosity about problems like this, ideally, with minimal guidance. Sometimes a problem is investigated through multiple meetings (and beyond).

1. Sixteen unit squares are arranged in a 4×4 square array. What is the maximum number of diagonals that can be drawn in these unit squares so that no two diagonals cross or share an endpoint?

2. Draw a few (non-crossing) diagonals in the unit squares of a 4×4 grid. Now assume that these diagonals are mirrors, and a ray of light enters the grid, and starts bouncing around. How long can its path be? (Count the length of the path as the number of unit squares you enter. If you enter a unit square twice, then count it twice.)

3. Consider a 4×4 grid again. Color some of the grid points red in such a way that no four red points would form the vertices of a rectangle (with sides parallel to the sides of the grid). What is the maximum number of grid points you can color?

Some questions, hints and references related to these problems are on the other side.

Questions, hints and references

1. (a) Consider square grids with sides 1,2,3, and 4. What pattern can you see? Does this pattern hold for squares of larger sizes?

   (b) Hint: There is a nice pattern for square grids with an even side.

   (c) What about rectangular grids? Can the argument for square grids with an even side be extended to cover the case of some rectangular grids?

   d) Can you devise an algorithm that would systematically check, and tell you what the maximum number of diagonals is?

   e) What happens to this problem in three dimensions?

   f) References: Online Encyclopedia of Integer Sequences, https://oeis.org/A264041, On the Maximum Number of Non-intersecting Diagonals, J. Integer Sequences, https://cs.uwaterloo.ca/journals/JIS/VOL20/Pinter/pinter3.html

2. (a) Reference: Jim Henle, The Proof and the Pudding: What Mathematicians, Cooks, and You Have in Common, Princeton, 2015; (first chapter available at http://press.princeton.edu/titles/10436.html)

3. (a) Mike Lawler's blog has nice videos on how kids think about this problem: https://mikesmathpage.wordpress.com/2015/07/09/

   (b) 1988 American Junior High School Math Exam, Problem 16

   (c) S. Fenner et.al., Rectangle Free Coloring of Grids, www.cs.umd.edu/~gasarch/papers/grid.pdf