PROBLEM SOLVING PARADIGMS FOR MATHEMATICAL RESEARCH

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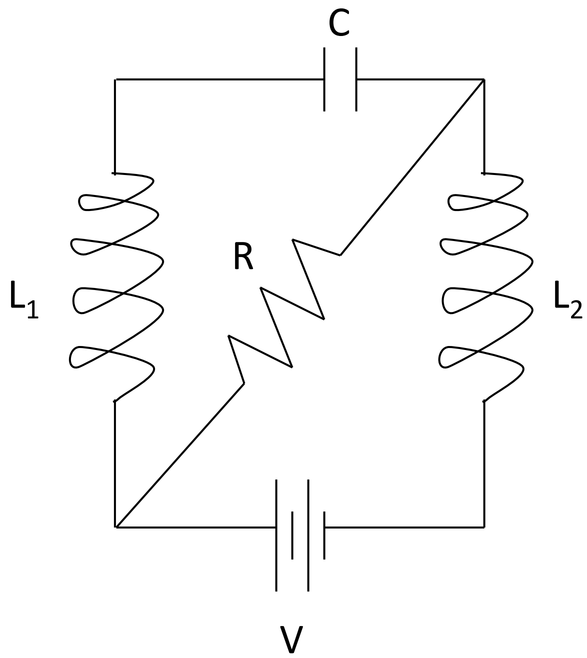
From problem solving to problem writing and research

- Inquiry-based explorations.
- Concrete observations and data collection.
- Pattern identification and formulation of conjectures.
- Counter-examples and refutations.
- Approximations and proofs.
- Articulating effective questions.

RLC ladders with non-trivial topology

- Existence and uniqueness of period.
- Dependence of period on number of loops.
- Dependence of period on topology of loops.

RLC ladders with non-trivial topology



$$\frac{L}{R}\dddot{I}_1 + 2\ddot{I}_1 + \frac{1}{RC}\dot{I}_1 + \frac{1}{LC}I_1 = 0$$

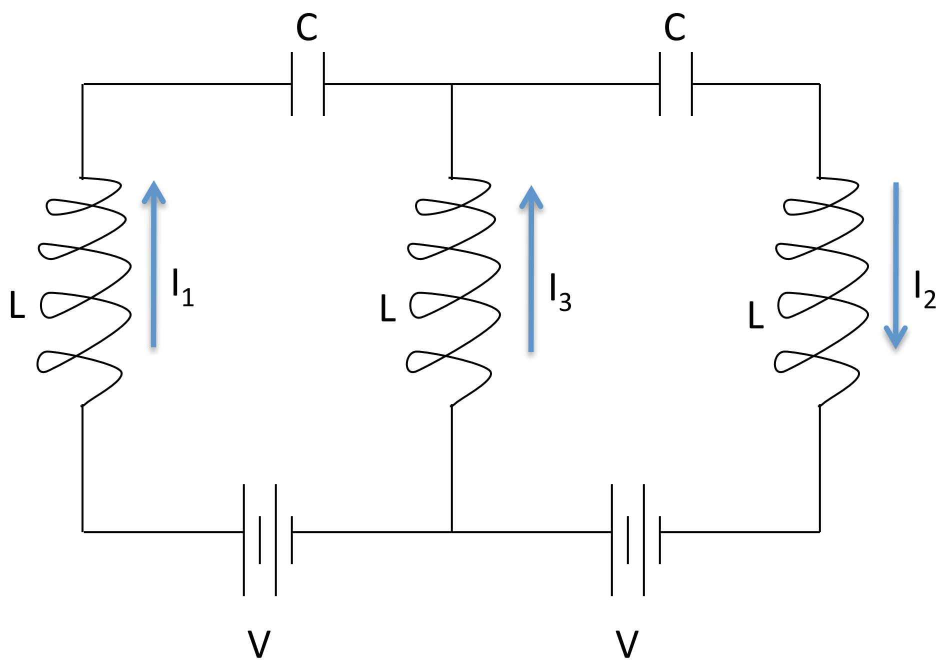
$$R \uparrow \infty \Rightarrow f = \frac{1}{\sqrt{2LC}}$$

$$R \downarrow 0 \Rightarrow f = \frac{1}{\sqrt{LC}}$$

$$0 < R < \infty \Rightarrow \quad \text{over - damped}$$

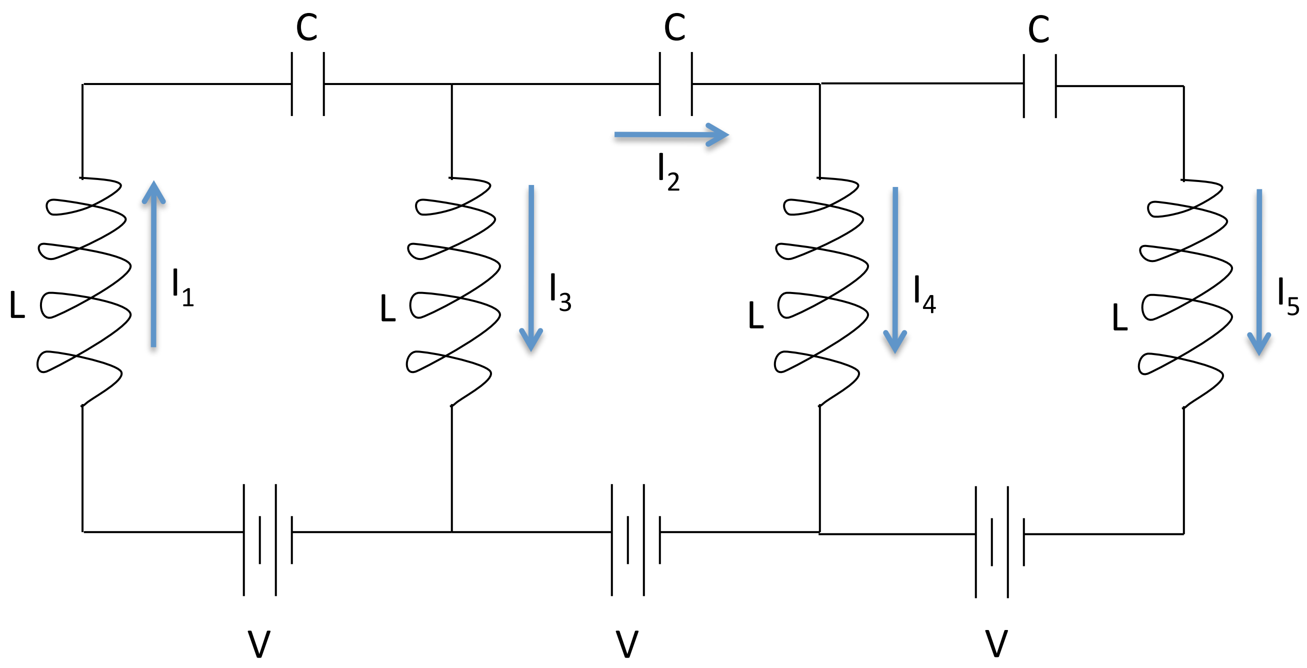
RLC ladders with non-trivial topology

$$\left.\begin{aligned} &\ddot{I}_3 + \frac{1}{3LC} I_3 = 0 \\ &\dot{I}_1(0) - \dot{I}_3(0) = \dot{I}_1(0) + 2\dot{I}_3(0) = \frac{V}{L} \end{aligned}\right\} \Rightarrow I_3 \equiv 0 \Rightarrow f = \frac{1}{\sqrt{LC}}$$



RLC ladders with non-trivial topology

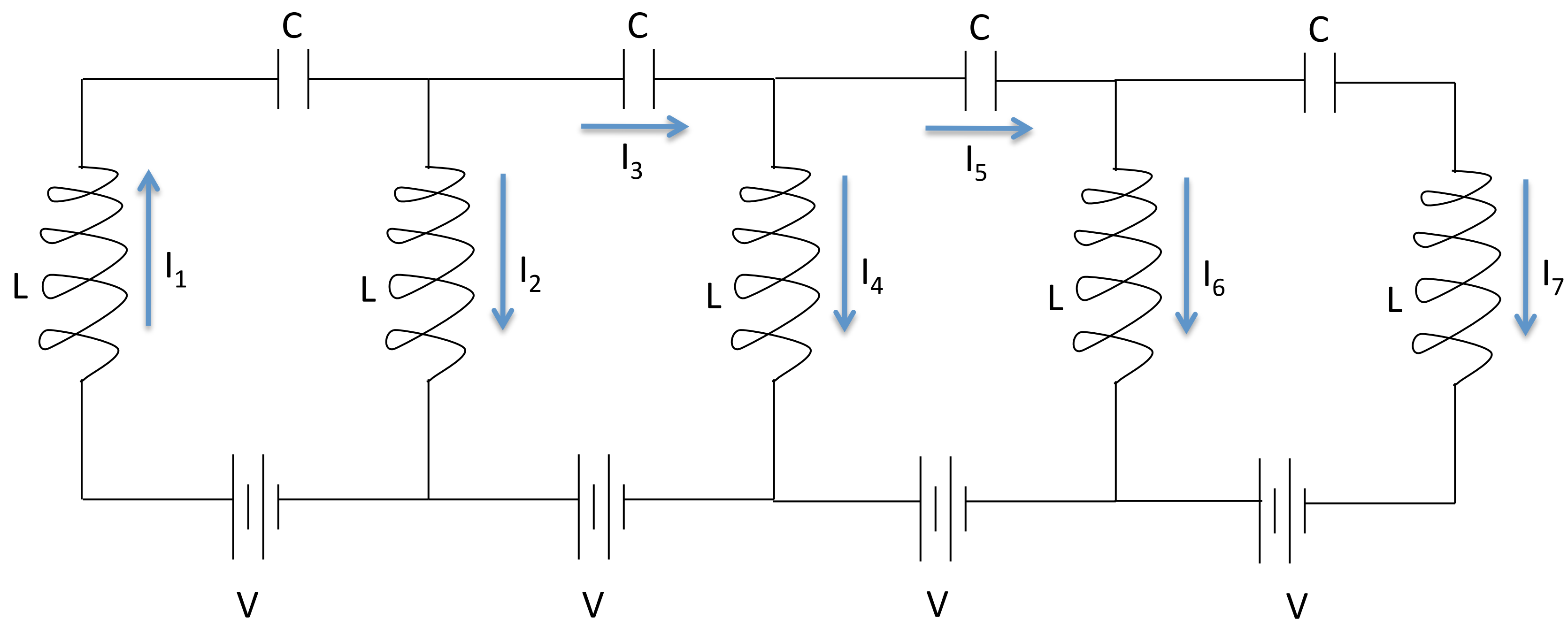
$$\left.\begin{aligned} 2LC\left(\ddot{I}_3+\ddot{I}_4\right)+I_3+I_4&=0\\ \dot{I}_3(0)-\dot{I}_4(0)&=\frac{V}{L}\end{aligned}\right\}\Rightarrow I_3, I_4\neq 0$$



RLC ladders with non-trivial topology

$$5L^2C^2I_4^{(4)} + 5LC\ddot{I}_4 + I_4 = 0 \Rightarrow$$

$$\Rightarrow f_1 = \sqrt{\frac{5+\sqrt{5}}{2LC}},\ f_2 = \sqrt{\frac{5-\sqrt{5}}{2LC}}$$



RLC ladders with non-trivial topology

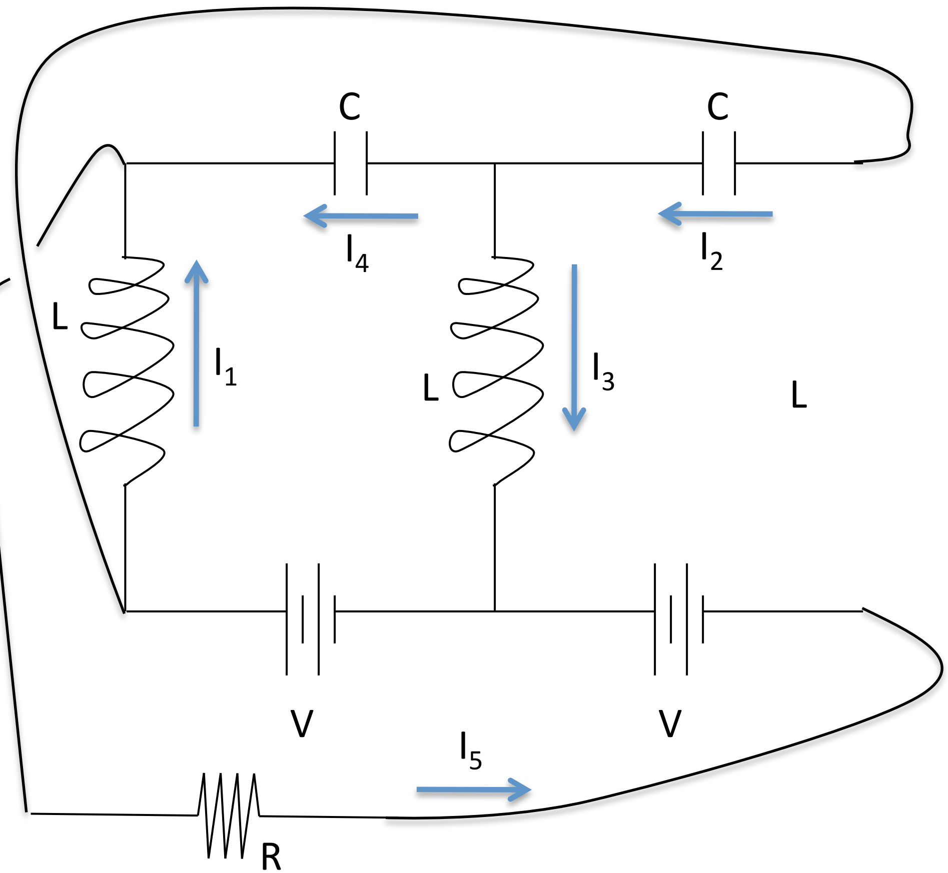
$$I_3^{(4)} + \frac{2}{RC} I_3^{(3)} + \frac{3}{LC}\ddot{I}_3 + \frac{1}{LC^2R}\dot{I}_3 + \frac{1}{L^2C^2} I_3 = 0$$

$$R \uparrow \infty \Rightarrow f_{1,2} = \sqrt{\frac{3 \pm \sqrt{5}}{2LC}}$$

$$R \downarrow 0 \Rightarrow f = \frac{1}{\sqrt{2LC}}$$

but I_1 is a short!

$$0 < R < \infty \Rightarrow \ ???$$



Exploring the trefoil knot and its variants

$$\left(\frac{x^2+y^2+z^2-5}{4}\right)^2+z^2=1$$

$$s\arctan\left(\frac{y}{x}\right)=2\arcsin z$$

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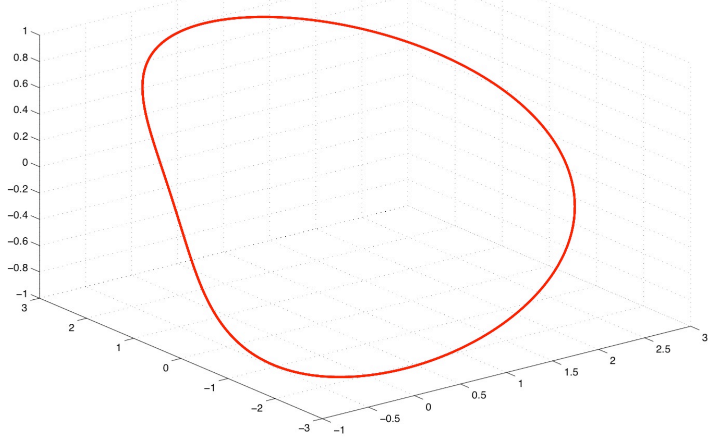
- Locate the extrema of the velocity vector and its radial and tangential components.

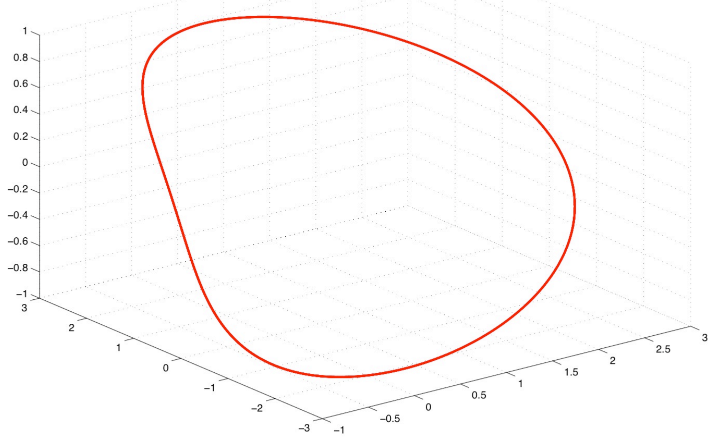
- Use curvature and torsion to determine the number of crossings.

Exploring the trefoil knot and its variants



$$s = 2$$

Exploring the trefoil knot and its variants



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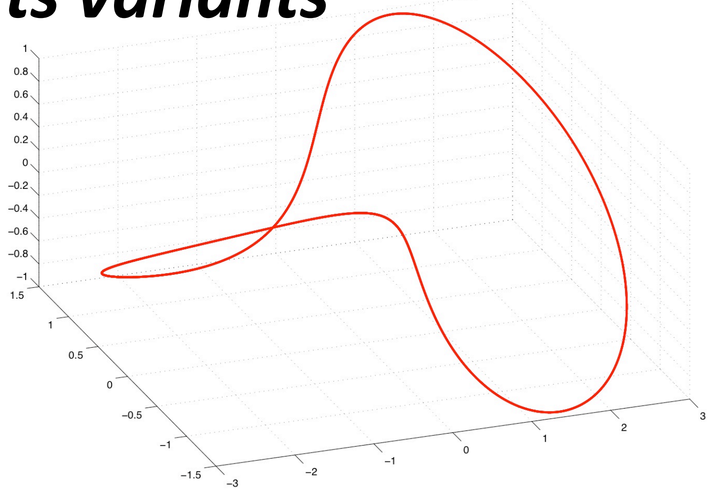
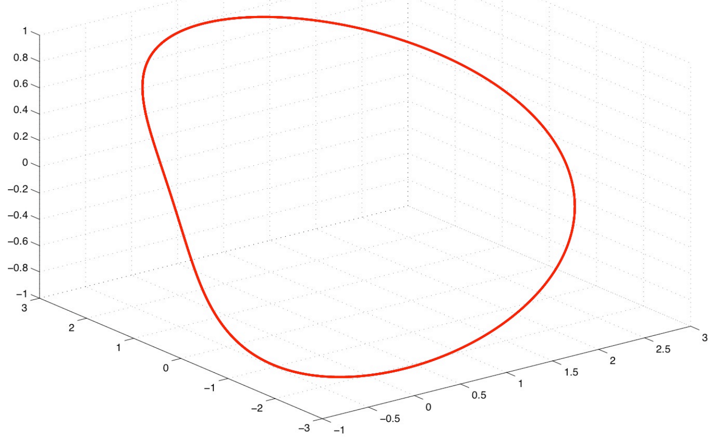


$s = 4$

Exploring the trefoil knot and its variants



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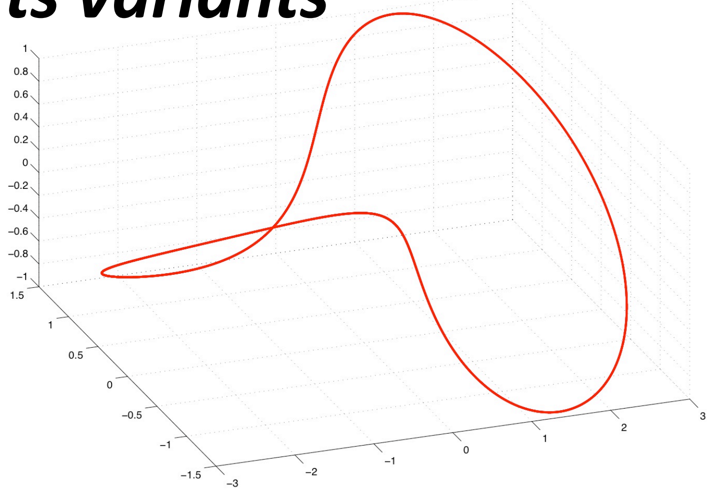

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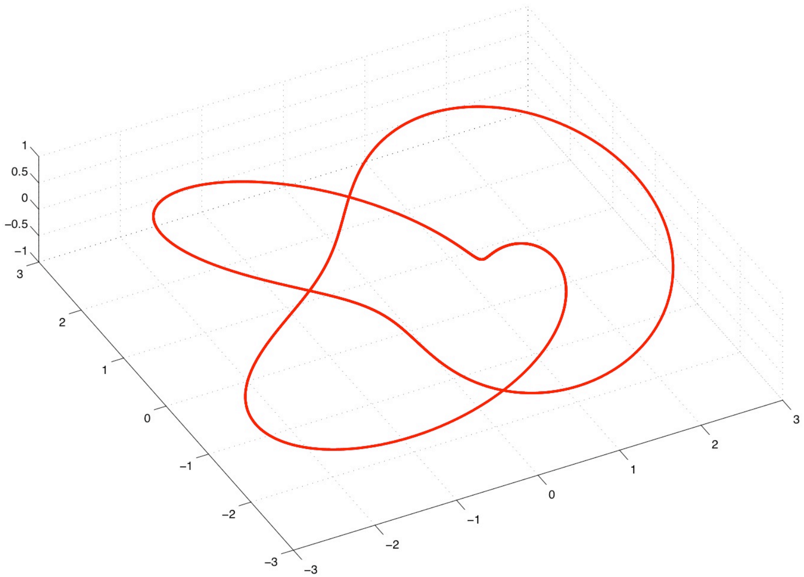
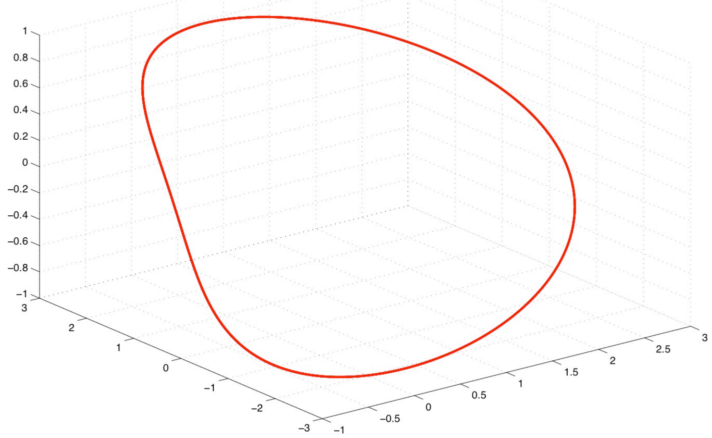
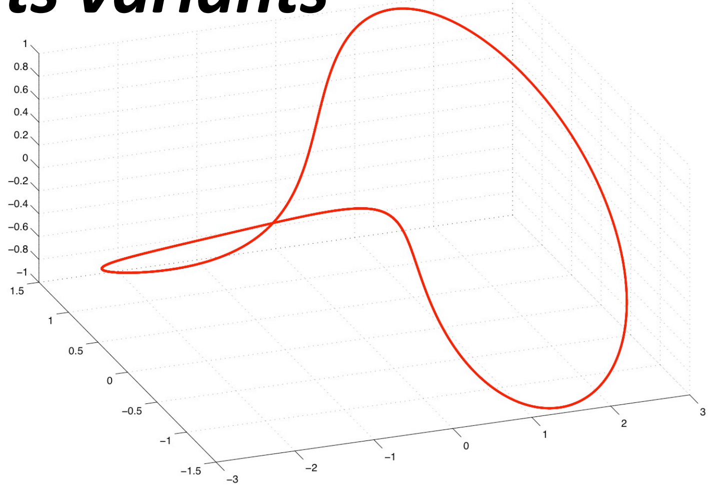


$s = 3$

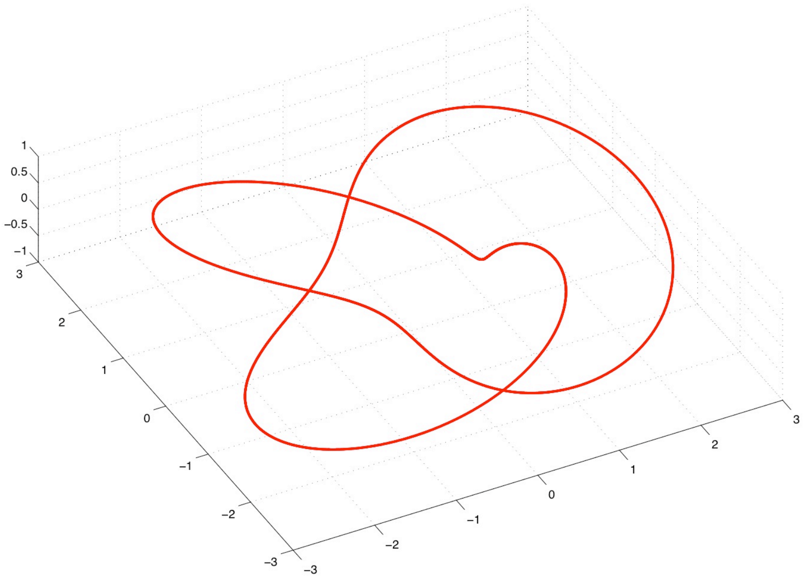
Exploring the trefoil knot and its variants



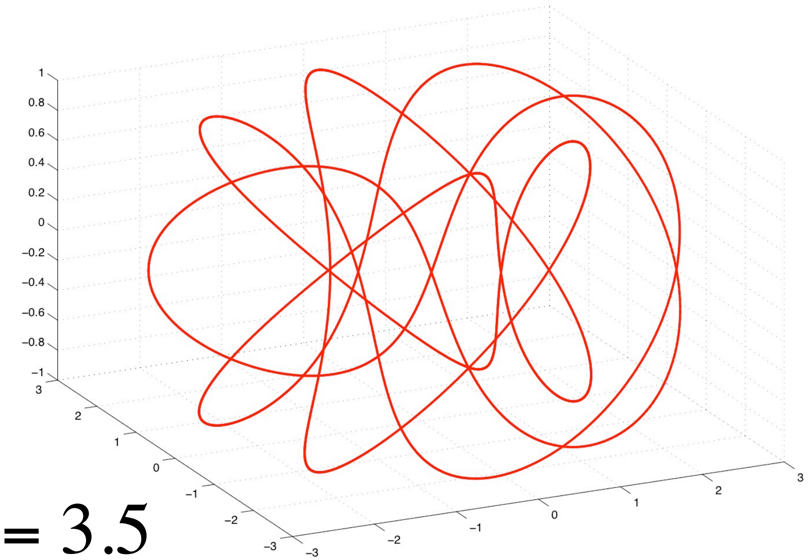
$s = 2$



$s = 4$

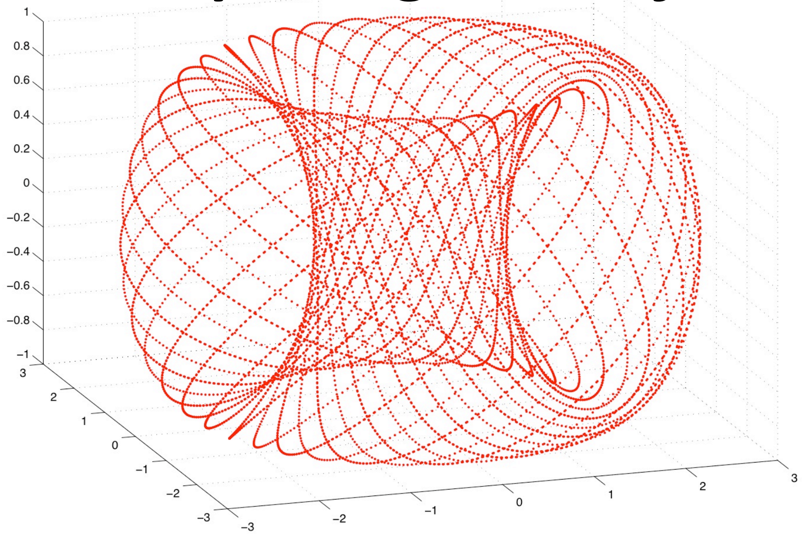


$s = 3$



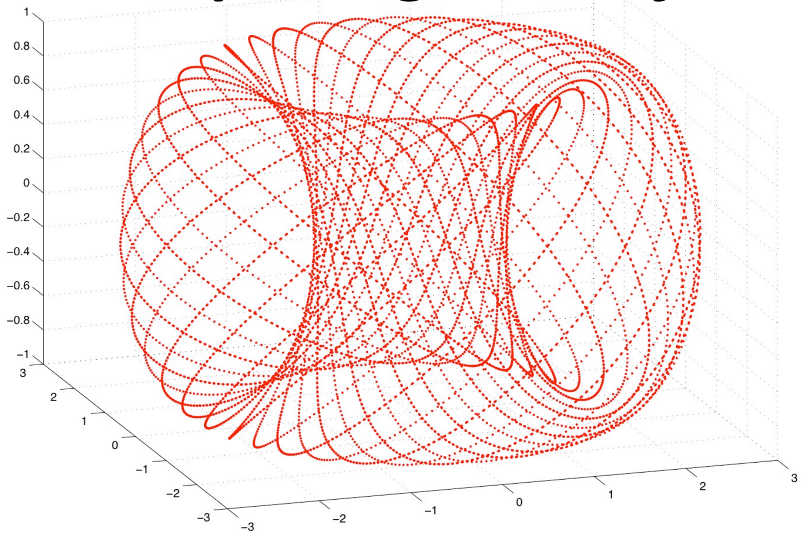
$s = 3.5$

Exploring the trefoil knot and its variants

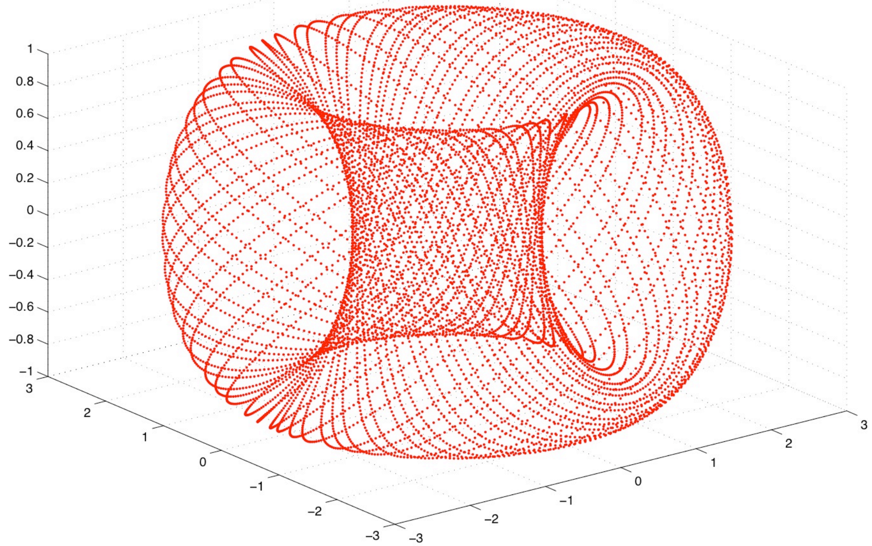


$s = 3.1$

Exploring the trefoil knot and its variants



$s = 3.1$



$s = \sqrt{10}$

Iterated functions on the complex plane

$$F(w,z) = w^z$$

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$$F^{(2)}(w,z) = F\big(w,F(w,z)\big) = w^{w^z}$$

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$$F^{(n+1)}(w,z) = F\left(w, F^{(n)}(w,z)\right)$$

Iterated functions on the complex plane

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$$f(c) = \lim_{n \to \infty} F^{(n)}(ic, i)$$

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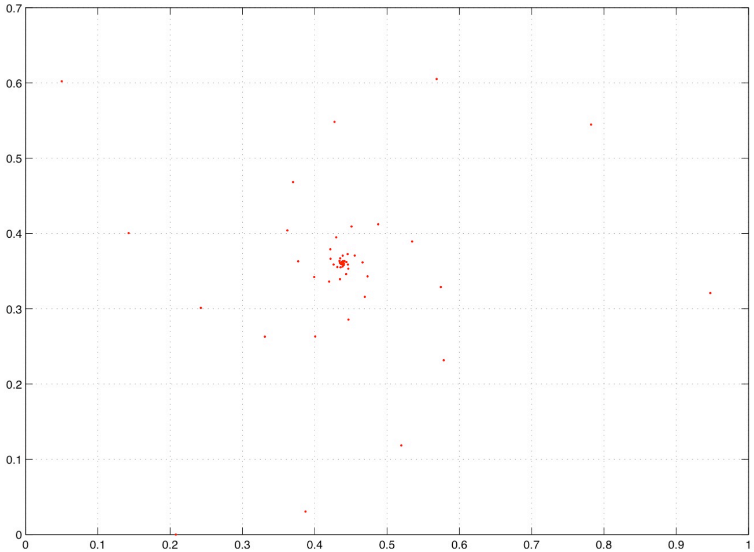
$$f(c) = \lim_{n \to \infty} F^{(n)}(ic, i)$$

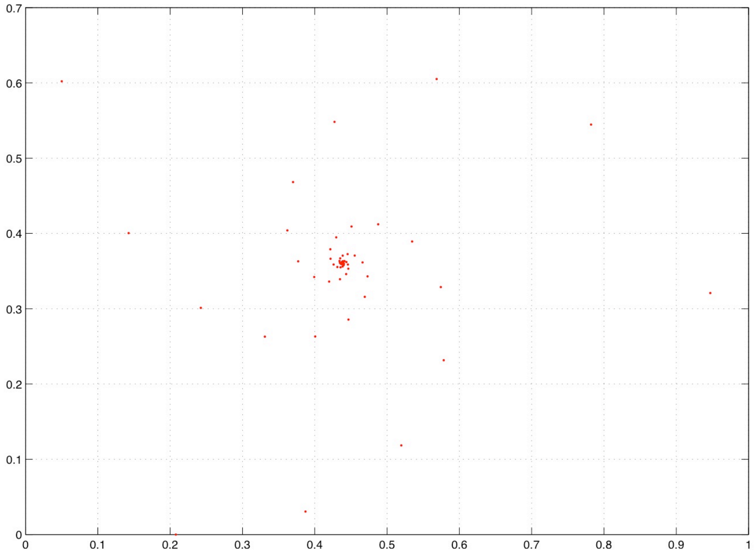
- When does f(c) exist?
- When is f(c) unique?
- What is f(c) equal to?
- How is f(c) approached?

Iterated functions on the complex plane

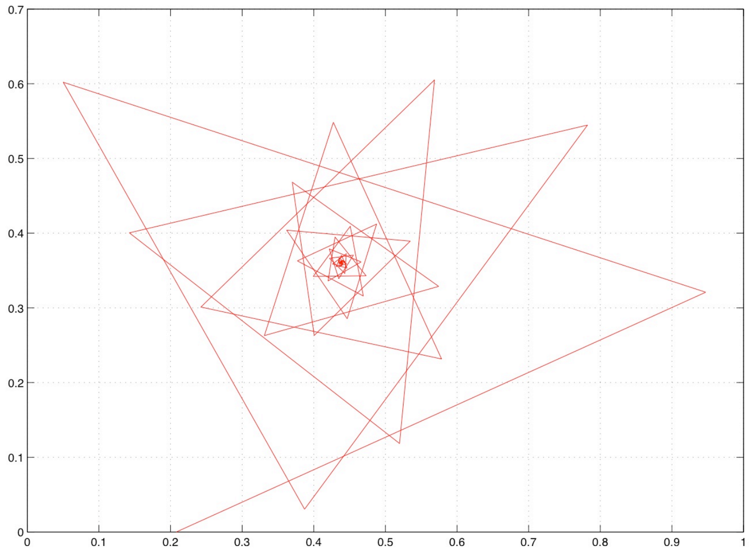


$$c = 1$$

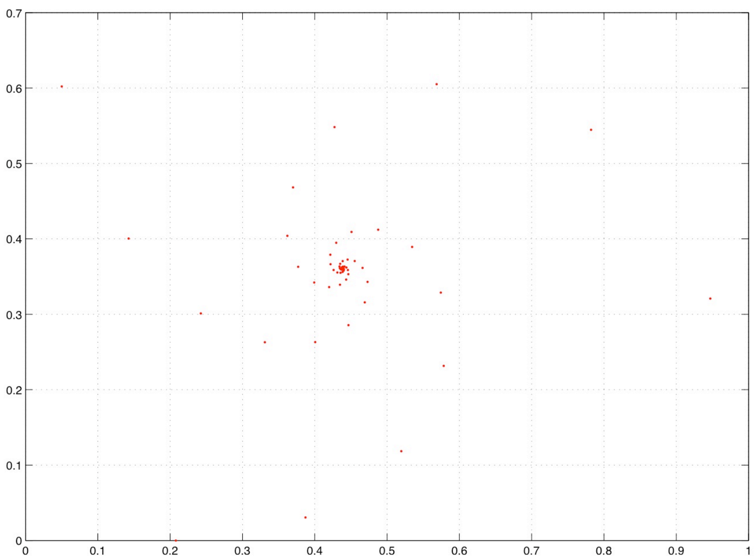
Iterated functions on the complex plane

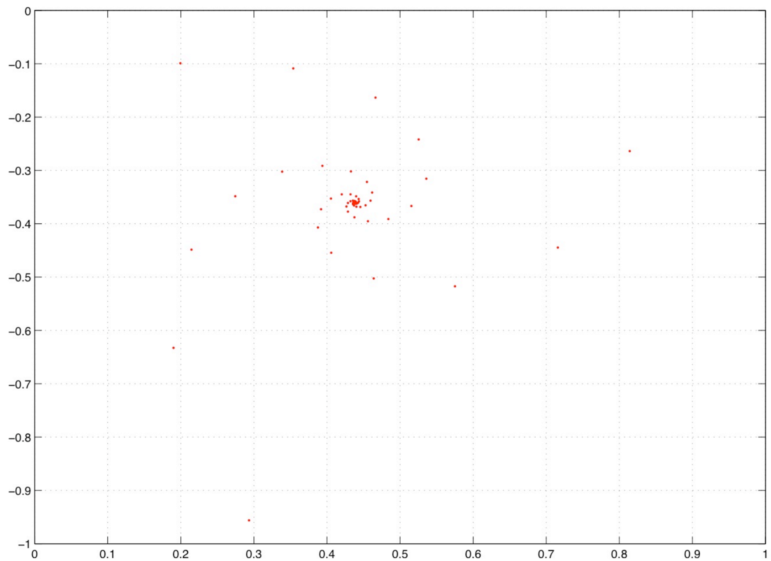


$$c = 1$$



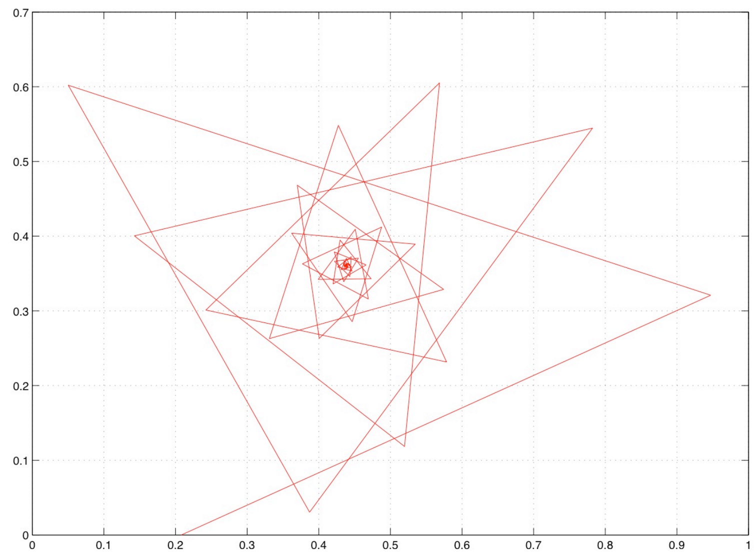
Iterated functions on the complex plane

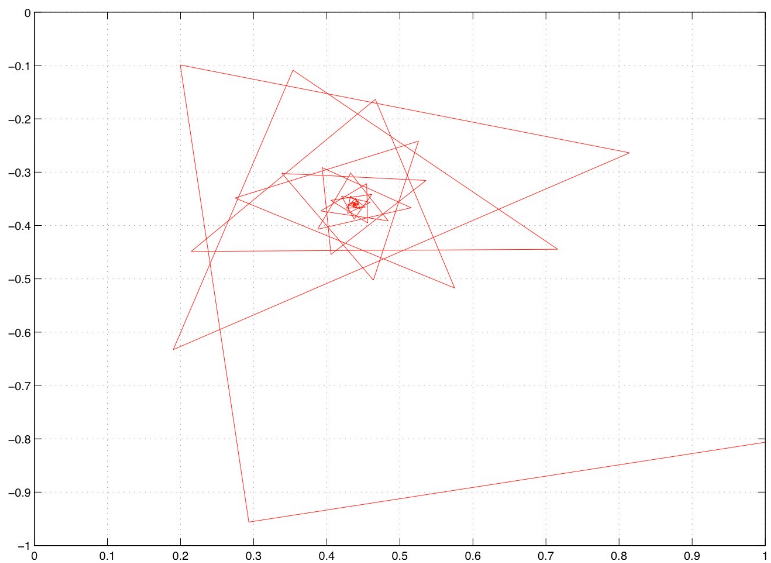




$c = 1$

$c = -1$





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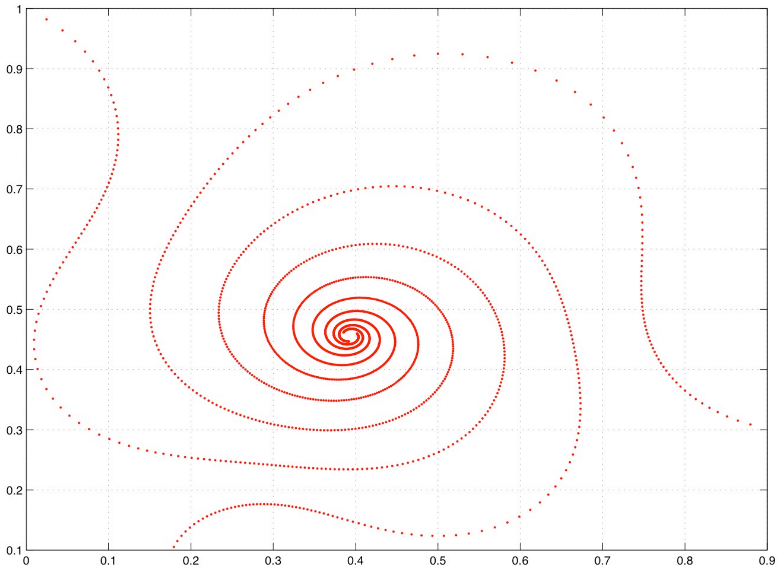
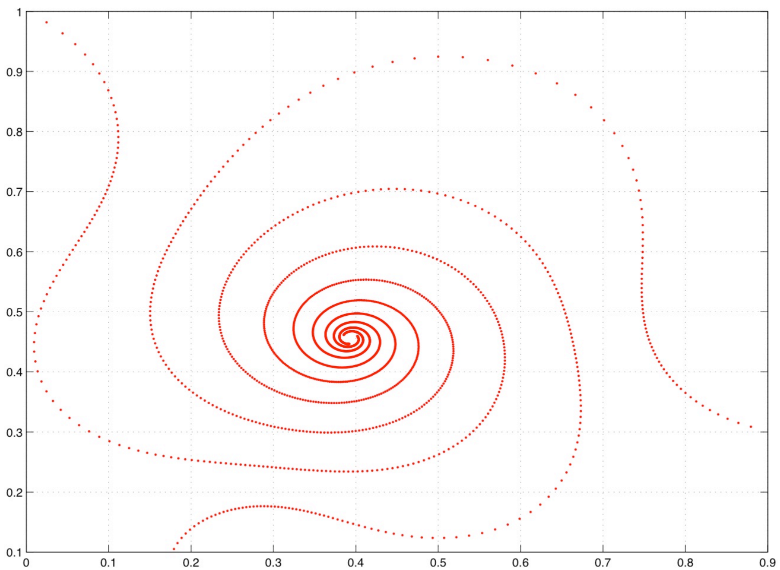
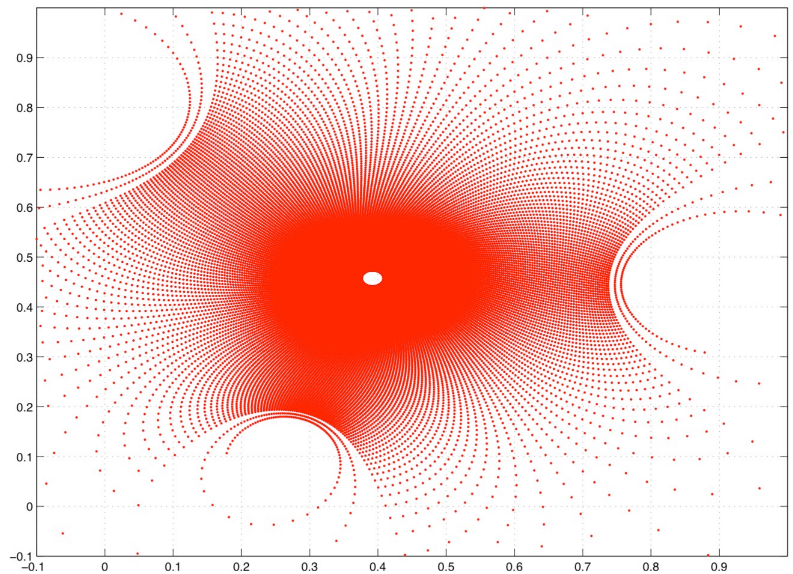


$c = 1.7$

Iterated functions on the complex plane

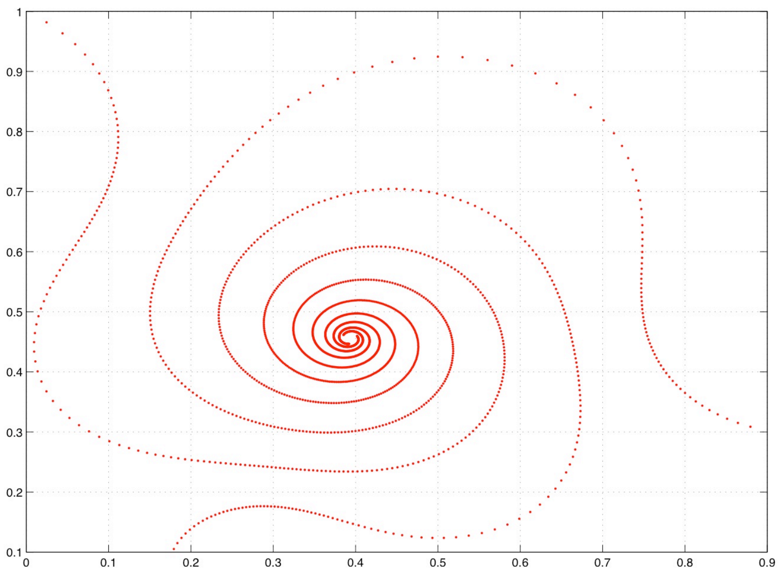


$c = 1.7$

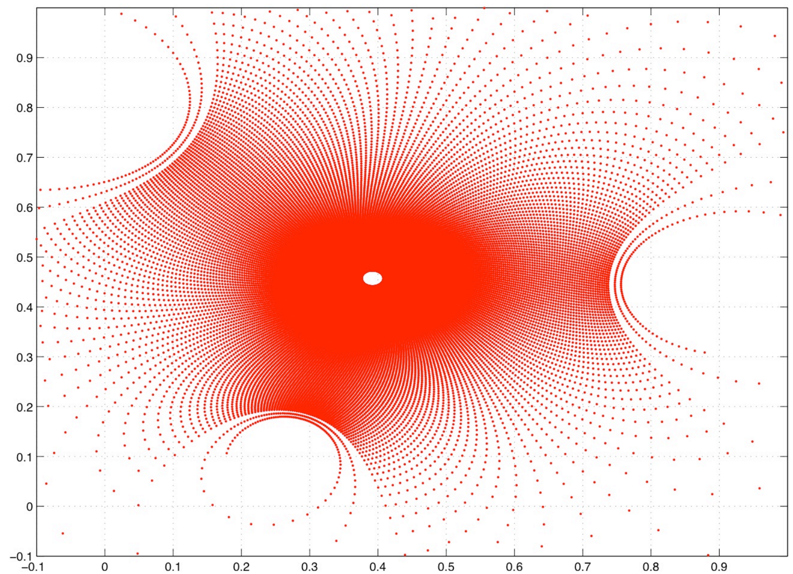


$c = 1.711$

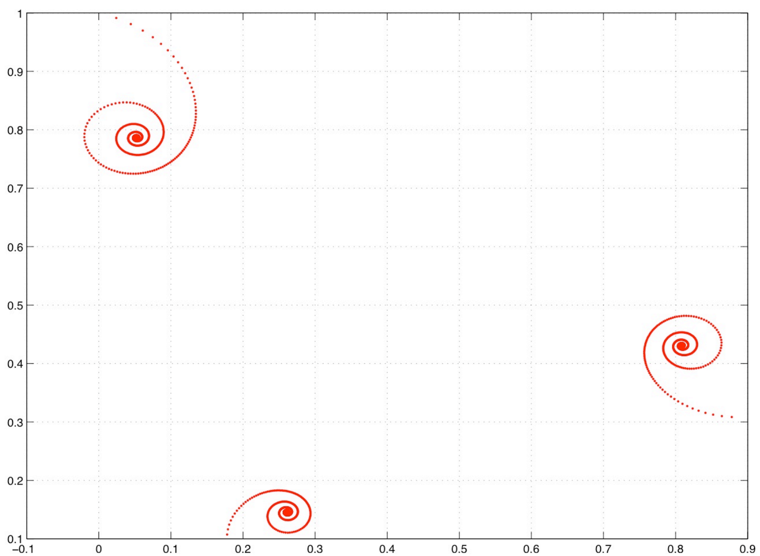
Iterated functions on the complex plane



$c = 1.7$



$c = 1.711$



$c = 1.72$

Configurations of points and lines

- Are there 100 lines on the plane that cross at exactly 1985 points?

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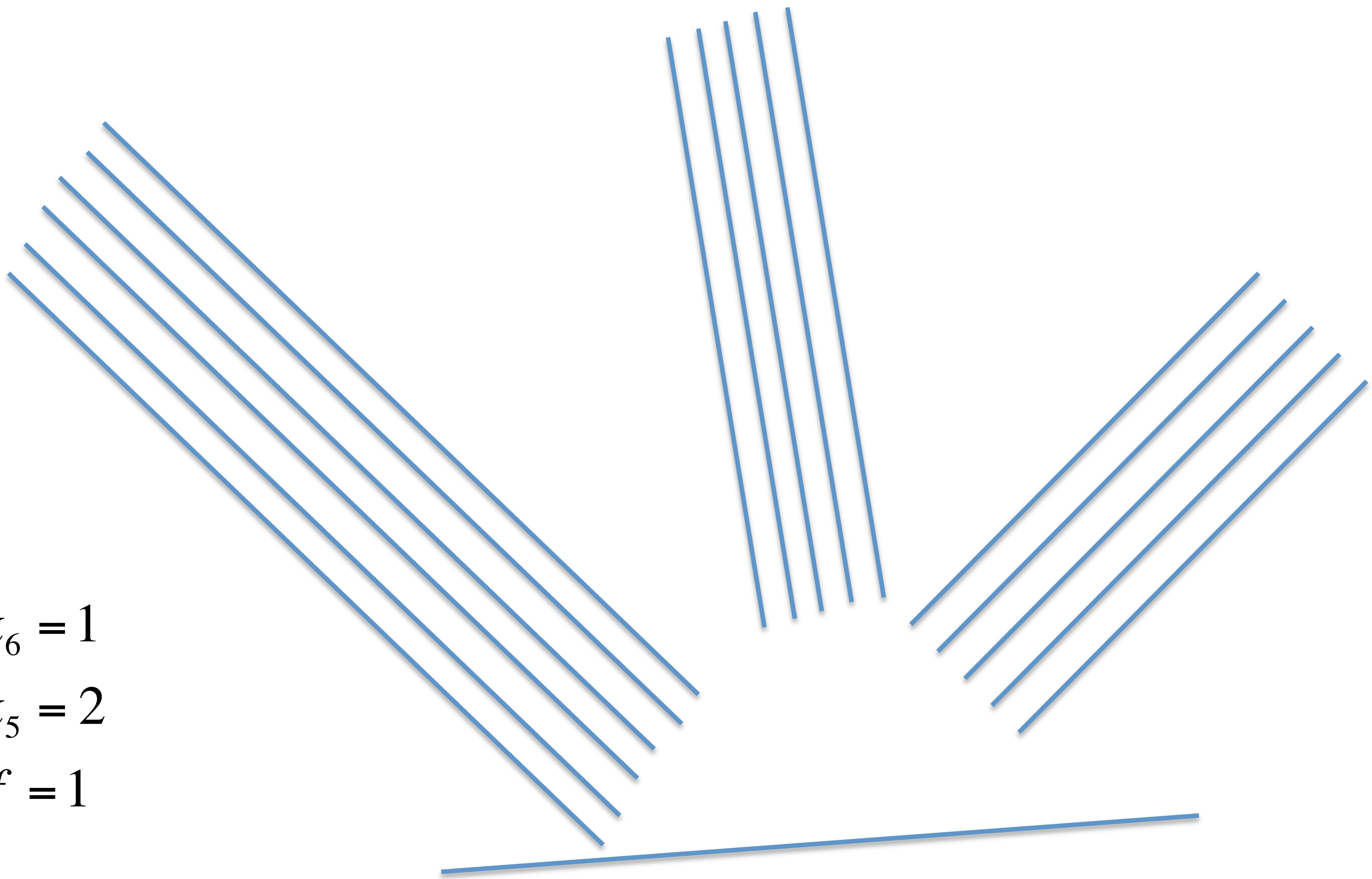
- Are there k lines on the plane that cross at exactly n points?

Configurations of points and lines

- Are there 100 lines on the plane that cross at exactly 1985 points?
- Are there 17 lines on the plane that cross at exactly 101 points?
- Are there k lines on the plane that cross at exactly n points?
- How many distinct such configurations are there?

Configurations of points and lines

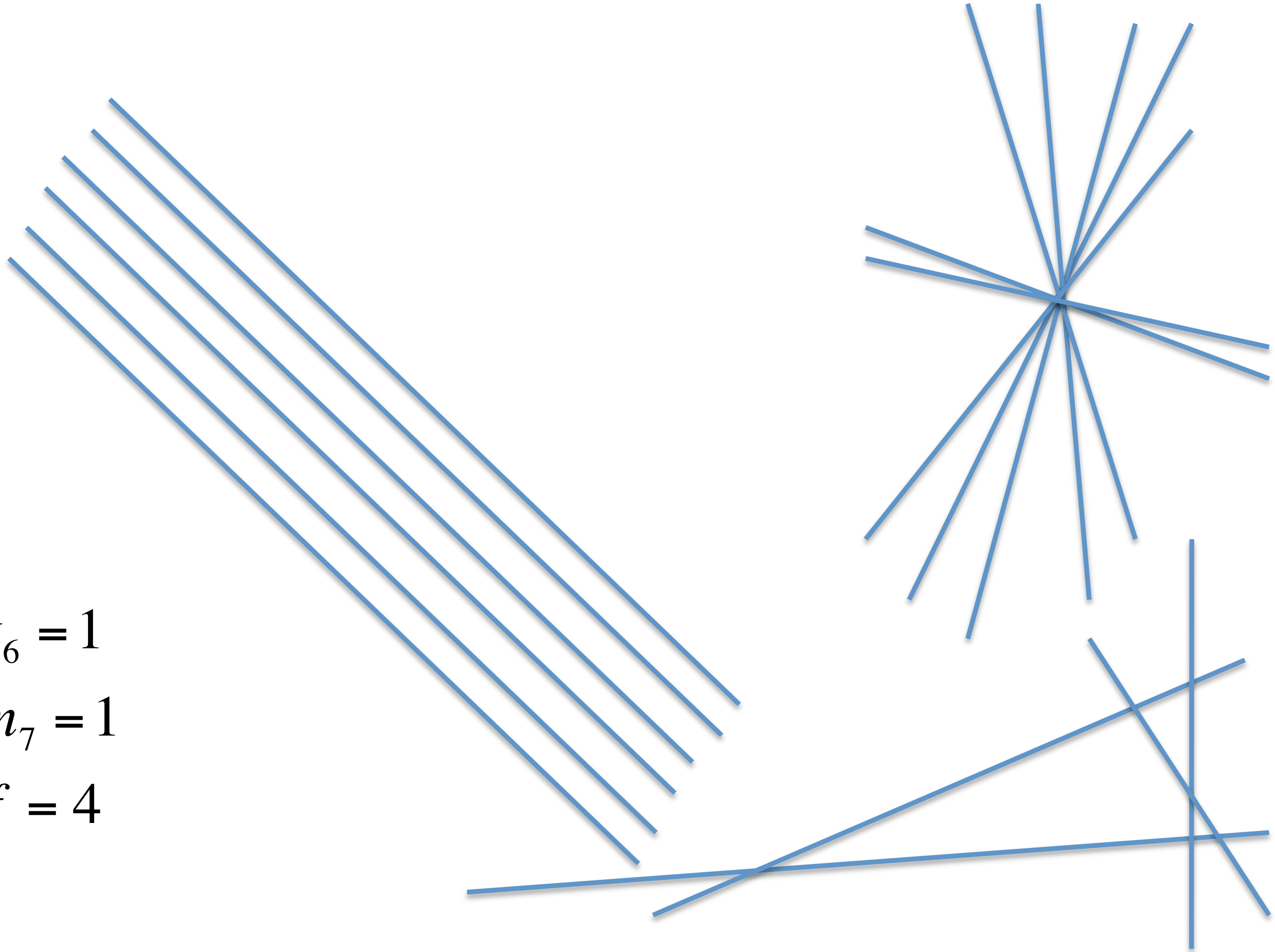


$$k_6 = 1$$

$$k_5 = 2$$

$$f = 1$$

Configurations of points and lines

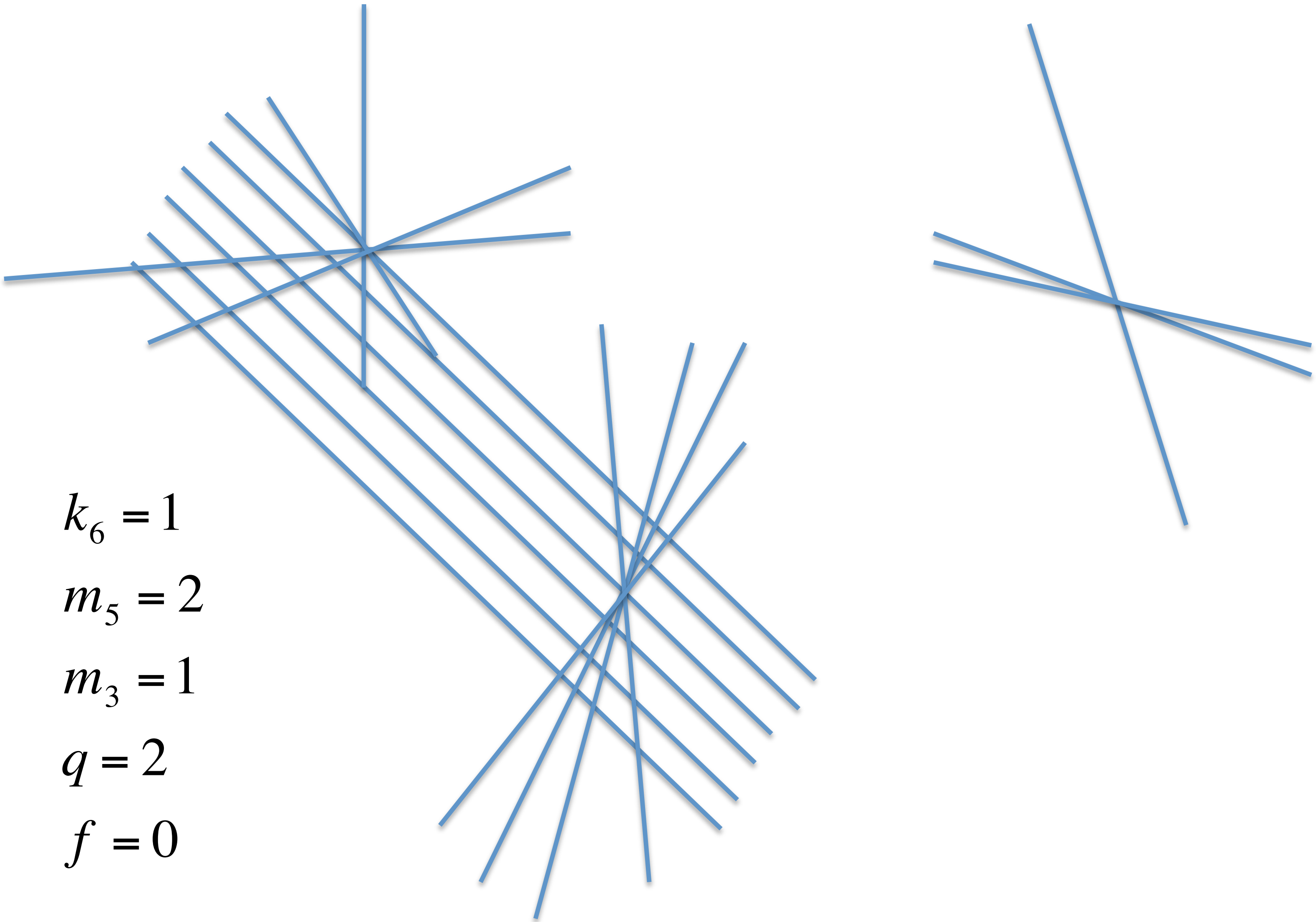


$$k_6 = 1$$

$$m_7 = 1$$

$$f = 4$$

Configurations of points and lines



$k_6 = 1$

$m_5 = 2$

$m_3 = 1$

$q = 2$

$f = 0$

Configurations of points and lines

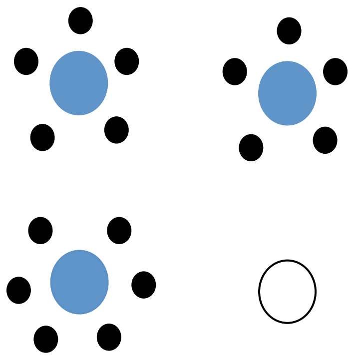
$$\binom{k}{2} - n = \sum m_i \left[\binom{i}{2} - 1 \right] + \sum k_j \binom{j}{2}$$

The shortfall in points is accounted for by the points lost to coincident lines plus the points lost to parallel lines.

$$k = \sum i m_i + \sum j k_j + f - q$$

The shortfall in lines is accounted for by free lines and overlaps.

Configurations of points and lines



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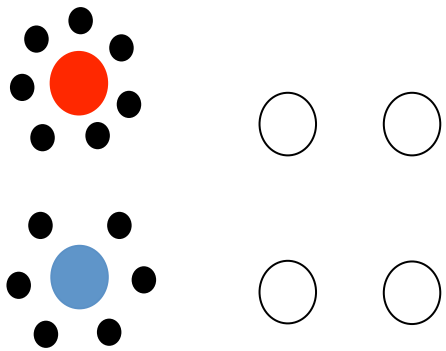

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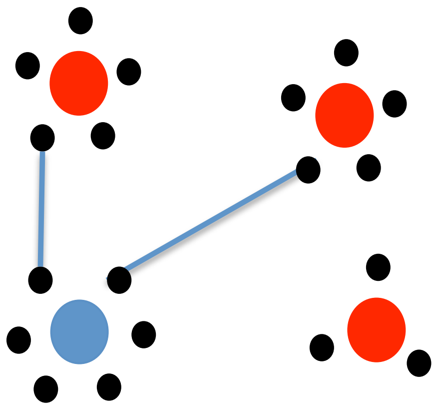
Configurations of points and lines



$k_6 = 1$

$m_5 = 2$

$m_3 = 1$

$q = 2$

$f = 0$

Multiplicity from the number of constrained "valence" graphs.

The reality game

- N players bet money out of their initial endowment on a coin toss.
- The probability of HEADS is equal to the proportion of money bet on HEADS.
- Winners split the pot equally among themselves, irrespective of their bets.
- The game is repeated with each player retaining their gains/losses.

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- Will one player always dominate in the long run?

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- Will one player always dominate in the long run?
- How long does it take before the winner is determined?
- What event instigates this choice?