Math Rumble Problems

American Mathematics Competitions

February 11, 2011

1. A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

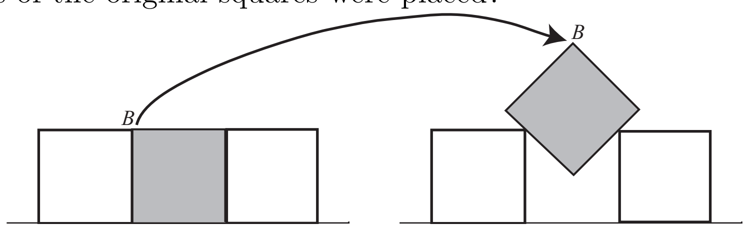
   Solution: The unit cubes have a total of $6n^3$ faces, of which $6n^2$ are red. Therefore
   $$\frac{1}{4} = \frac{6n^2}{6n^3} = \frac{1}{n}, \quad \text{so} \quad n = 4.$$

2. How many positive integers n satisfy the following condition:
   $$(130n)^{50} > n^{100} > 2^{200}\ ?$$

   Solution: The condition is equivalent to
   $$130n > n^2 > 2^4 = 16, \quad \text{so} \quad 130n > n^2 \text{ and } n^2 > 16.$$

   This implies that $130 > n > 4$. So n can be any of the 125 integers strictly between 130 and 4.

3. Three one-inch squares are placed with their bases on a line. The center square is lifted out and rotated 45°, as shown. Then it is centered and lowered into its original location until it touches both of the adjoining squares. How many inches is the point B from the line on which the bases of the original squares were placed?

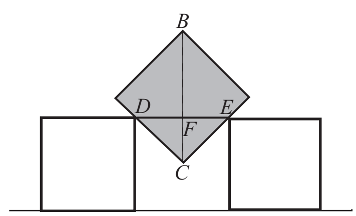


Solution: Consider the rotated middle square shown in the figure. It will drop until length DE is 1 inch. Thus

$$FC = DF = FE = \frac{1}{2} \quad \text{and} \quad BC = \sqrt{2}.$$

Hence $BF = \sqrt{2} - 1/2$. This is added to the 1 inch height of the supporting squares, so the overall height of point B above the line is

$$1 + BF = \sqrt{2} + \frac{1}{2} \text{ inches.}$$



4. One fair die has faces 1, 1, 2, 2, 3, 3 and another has faces 4, 4, 5, 5, 6, 6. The dice are rolled and the numbers on the top faces are added. What is the probability that the sum will be odd?

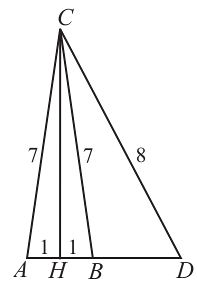
   Solution: An odd sum requires either that the first die is even and the second is odd or that the first die is odd and the second is even. The probability is
   $$\frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}.$$

5. In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is BD?

   Solution: Let $\overline{CH}$ be an altitude of $\triangle ABC$. Applying the Pythagorean Theorem to $\triangle CHB$ and to $\triangle CHD$ produces

   $$8^2 - (BD + 1)^2 = CH^2 = 7^2 - 1^2 = 48, \quad \text{so} \quad (BD + 1)^2 = 16.$$

   Thus $BD = 3$.



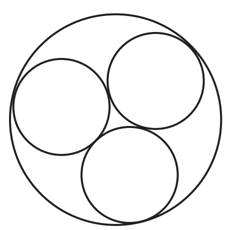
6. Patty has 20 coins consisting of nickels and dimes. If her nickels were dimes and her dimes were nickels, she would have 70 cents more. How much are her coins worth?

   Solution: Because the value of Patty's money would increase if the dimes and nickels were interchanged, she must have more nickels than dimes. Interchanging one nickel for a dime increases the amount by 5 cents, so she has $70/5 = 14$ more nickels than dimes. Therefore she has

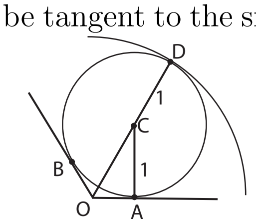
   $$\frac{1}{2}(20-14) = 3 \quad \text{dimes} \quad \text{and} \quad 20-3=17 \quad \text{nickels},$$

   and her coins are worth $3 \cdot 10 + 17 \cdot 5 = 115$ cents $= \$1.15$.

7. Three circles of radius 1 are externally tangent to each other and internally tangent to a larger circle. What is the radius of the large circle?



   Solution: Let O be the center of the large circle, let C be the center of one of the small circles, and let $\overline{OA}$ and $\overline{OB}$ be tangent to the small circle at A and B.



By symmetry, $\angle AOB = 120^\circ$ and $\angle AOC = 60^\circ$. Thus $\triangle AOC$ is a 30-60-90 degree right triangle, and $AC = 1$, so

$$OC = \frac{2}{\sqrt{3}}AC = \frac{2\sqrt{3}}{3}.$$

If OD is a radius of the large circle through C, then

$$OD = CD + OC = 1 + \frac{2\sqrt{3}}{3} = \frac{3 + 2\sqrt{3}}{3}.$$

8. Henry's Hamburger Heaven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered?

   Solution: A customer makes one of two choices for each condiment, to include it or not to include it. The choices are made independently, so there are $2^8 = 256$ possible combinations of condiments. For each of those combinations there are three choices regarding the number of meat patties, so there are altogether $(3)(256) = 768$ different kinds of hamburger.

9. Given that $-4 \le x \le -2$ and $2 \le y \le 4$, what is the largest possible value of $(x + y)/x$?

   Solution: Because

   $$\frac{x+y}{x} = 1 + \frac{y}{x} \quad \text{and} \quad \frac{y}{x} < 0,$$

   the value is maximized when $|y/x|$ is minimized, that is, when $|y|$ is minimized and $|x|$ is maximized. So $y = 2$ and $x = -4$ gives the largest value, which is $1 + (-1/2) = 1/2$.

10. Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

    Solution: The result will occur when both A and B have either 0, 1, 2, or 3 heads, and these probabilities are shown in the table.

Heads	0	1	2	3
A	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
B	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$

The probability of both coins having the same number of heads is

$$\frac{1}{8} \cdot \frac{1}{16} + \frac{3}{8} \cdot \frac{4}{16} + \frac{3}{8} \cdot \frac{6}{16} + \frac{1}{8} \cdot \frac{4}{16} = \frac{35}{128}.$$