

# Math Wrangle Practice Problems

## American Mathematics Competitions

November 19, 2010

1. Find the sum of all positive two-digit integers that are divisible by each of their digits.
2. A finite set  $\mathcal{S}$  of distinct real numbers has the following properties: the mean of  $\mathcal{S} \cup \{1\}$  is 13 less than the mean of  $\mathcal{S}$ , and the mean of  $\mathcal{S} \cup \{2001\}$  is 27 more than the mean of  $\mathcal{S}$ . Find the mean of  $\mathcal{S}$ .
3. Find the sum of all the roots, real and non-real, of the equation  $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$ , given that there are no multiple roots.
4. In triangle  $ABC$ , angles  $A$  and  $B$  measure 60 degrees and 45 degrees, respectively. The bisector of angle  $A$  intersects  $\overline{BC}$  at  $T$ , and  $AT = 24$ . The area of the triangle  $ABC$  can be written in the form  $a + b\sqrt{c}$ , where  $a$ ,  $b$ , and  $c$  are positive integers, and  $c$  is not divisible by the square of any prime. Find  $a + b + c$ .
5. An equilateral triangle is inscribed in the ellipse whose equation is  $x^2 + 4y^2 = 4$ . One vertex of the triangle is  $(0, 1)$ , one altitude is contained in the  $y$ -axis, and the length of each side is  $\sqrt{\frac{m}{n}}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
6. A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
7. Triangle  $ABC$  has  $AB = 21$ ,  $AC = 22$ , and  $BC = 20$ . Points  $D$  and  $E$  are located on  $\overline{AB}$  and  $\overline{AC}$ , respectively, such that  $\overline{DE}$  is parallel to  $\overline{BC}$  and contains the center of the inscribed circle of triangle  $ABC$ .

Then  $DE = m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

8. Call a positive integer  $N$  a *7-10 double* if the digits of the base-7 representation of  $N$  form a base-10 number that is twice  $N$ . For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?
9. In triangle  $ABC$ ,  $AB = 13$ ,  $BC = 15$  and  $CA = 17$ . Point  $D$  is on  $\overline{AB}$ ,  $E$  is on  $\overline{BC}$ , and  $F$  is on  $\overline{CA}$ . Let  $AD = p \cdot AB$ ,  $BE = q \cdot BC$ , and  $CF = r \cdot CA$ , where  $p$ ,  $q$ , and  $r$  are positive and satisfy  $p + q + r = 2/3$  and  $p^2 + q^2 + r^2 = 2/5$ . The ratio of the area of triangle  $DEF$  to the area of triangle  $ABC$  can be written in the form  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
10. Let  $\mathcal{S}$  be the set of points whose coordinates  $x$ ,  $y$ , and  $z$  are integers that satisfy  $0 \leq x \leq 2$ ,  $0 \leq y \leq 3$ , and  $0 \leq z \leq 4$ . Two distinct points are randomly chosen from  $\mathcal{S}$ . The probability that the midpoint of the segment they determine also belongs to  $\mathcal{S}$  is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
11. In a rectangular array of points, with 5 rows and  $N$  columns, the points are numbered consecutively from left to right beginning with the top row. Thus the top row is numbered 1 through  $N$ , the second row is numbered  $N + 1$  through  $2N$ , and so forth. Five points,  $P_1, P_2, P_3, P_4$ , and  $P_5$ , are selected so that each  $P_i$  is in row  $i$ . Let  $x_i$  be the number associated with  $P_i$ . Now renumber the array consecutively from top to bottom, beginning with the first column. Let  $y_i$  be the number associated with  $P_i$  after renumbering. It is found that  $x_1 = y_2$ ,  $x_2 = y_1$ ,  $x_3 = y_4$ ,  $x_4 = y_5$ , and  $x_5 = y_3$ . Find the smallest possible value of  $N$ .
12. A sphere is inscribed in the tetrahedron whose vertices are  $A = (6, 0, 0)$ ,  $B = (0, 4, 0)$ ,  $C = (0, 0, 2)$ , and  $D = (0, 0, 0)$ . The radius of the sphere is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
13. In a certain circle, the chord of a  $d$ -degree arc is 22 centimeters long, and the chord of a  $2d$ -degree arc is 20 centimeters longer than the chord of a  $3d$ -degree arc, where  $d < 120$ . The length of the chord of

a  $3d$ -degree arc is  $-m + \sqrt{n}$  centimeters, where  $m$  and  $n$  are positive integers. Find  $m + n$ .

14. A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?
15. The numbers 1, 2, 3, 4, 5, 6, 7, and 8 are randomly written on the faces of a regular octahedron so that each face contains a different number. The probability that no two consecutive numbers, where 8 and 1 are considered to be consecutive, are written on faces that share an edge is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
16. Given that
- $x$  and  $y$  are both integers between 100 and 999, inclusive;
  - $y$  is the number formed by reversing the digits of  $x$ ; and
  - $z = |x - y|$ .

How many distinct values of  $z$  are possible?

17. Three of the vertices of a cube are  $P = (7, 12, 10)$ ,  $Q = (8, 8, 1)$ , and  $R = (11, 3, 9)$ . What is the surface area of the cube?
18. It is given that  $\log_6 a + \log_6 b + \log_6 c = 6$ , where  $a$ ,  $b$ , and  $c$  are positive integers that form an increasing geometric sequence and  $b - a$  is the square of an integer. Find  $a + b + c$ .
19. Find the sum of all positive integers  $a = 2^n 3^m$ , where  $n$  and  $m$  are non-negative integers, for which  $a^6$  is not a divisor of  $6^a$ .
20. Find the integer that is closest to  $1000 \sum_{n=3}^{10000} \frac{1}{n^2 - 4}$ .
21. It is known that, for all positive integers  $k$ ,

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Find the smallest positive integer  $k$  such that  $1^2 + 2^2 + 3^2 + \cdots + k^2$  is a multiple of 200.

22. Find the least positive integer  $k$  for which the equation  $\left\lfloor \frac{2002}{n} \right\rfloor = k$  has no integer solutions for  $n$ . (The notation  $\lfloor x \rfloor$  means the greatest integer less than or equal to  $x$ .)
23. Let  $\mathcal{S}$  be the set  $\{1, 2, 3, \dots, 10\}$ . Let  $n$  be the number of sets of two non-empty disjoint subsets of  $\mathcal{S}$ . (*Disjoint sets* are defined as sets that have no common elements.) Find the remainder obtained when  $n$  is divided by 1000.
24. While finding the sine of a certain angle, an absent-minded professor failed to notice that his calculator was not in the correct angular mode. He was lucky to get the right answer. The two least positive real values of  $x$  for which the sine of  $x$  degrees is the same as the sine of  $x$  radians are  $\frac{m\pi}{n - \pi}$  and  $\frac{p\pi}{q + \pi}$ , where  $m$ ,  $n$ ,  $p$  and  $q$  are positive integers. Find  $m + n + p + q$ .
25. Two distinct, real, infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is  $1/8$ , and the second term of both series can be written in the form  $\frac{\sqrt{m} - n}{p}$ , where  $m$ ,  $n$ , and  $p$  are positive integers and  $m$  is not divisible by the square of any prime. Find  $100m + 10n + p$ .
26. A basketball player has a constant probability of .4 of making any given shot, independent of previous shots. Let  $a_n$  be the ratio of shots made to shots attempted after  $n$  shots. The probability that  $a_{10} = .4$  and  $a_n \leq .4$  for all  $n$  such that  $1 \leq n \leq 9$  is given to be  $p^a q^b r / (s^c)$ , where  $p$ ,  $q$ ,  $r$ , and  $s$  are primes, and  $a$ ,  $b$ , and  $c$  are positive integers. Find  $(p + q + r + s)(a + b + c)$ .
27. In triangle  $ABC$ , point  $D$  is on  $\overline{BC}$  with  $CD = 2$  and  $DB = 5$ , point  $E$  is on  $\overline{AC}$  with  $CE = 1$  and  $EA = 3$ ,  $AB = 8$ , and  $\overline{AD}$  and  $\overline{BE}$  intersect at  $P$ . Points  $Q$  and  $R$  lie on  $\overline{AB}$  so that  $\overline{PQ}$  is parallel to  $\overline{CA}$  and  $\overline{PR}$  is parallel to  $\overline{CB}$ . It is given that the ratio of the area of triangle  $PQR$  to the area of triangle  $ABC$  is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

28. The perimeter of triangle  $APM$  is 152, and angle  $PAM$  is a right angle. A circle of radius 19 with center  $O$  on  $\overline{AP}$  is drawn so that it is tangent to  $\overline{AM}$  and  $\overline{PM}$ . Given that  $OP = m/n$ , where  $m$  and  $n$  are relatively prime positive integers, find  $m + n$ .
29. Circles  $\mathcal{C}_1$  and  $\mathcal{C}_2$  intersect at two points, one of which is  $(9, 6)$ , and the product of their radii is 68. The  $x$ -axis and the line  $y = mx$ , where  $m > 0$ , are tangent to both circles. It is given that  $m$  can be written in the form  $a\sqrt{b}/c$ , where  $a$ ,  $b$ , and  $c$  are positive integers,  $b$  is not divisible by the square of any prime, and  $a$  and  $c$  are relatively prime. Find  $a + b + c$ .
30. Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
31. Jane is 25 years old. Dick is older than Jane. In  $n$  years, where  $n$  is a positive integer, Dick's age and Jane's age will both be two-digit numbers and will have the property that Jane's age is obtained by interchanging the digits of Dick's age. Let  $d$  be Dick's present age. How many ordered pairs of positive integers  $(d, n)$  are possible?
32. Consider the sequence defined by  $a_k = \frac{1}{k^2 + k}$  for  $k \geq 1$ . Given that  $a_m + a_{m+1} + \cdots + a_{n-1} = 1/29$ , for positive integers  $m$  and  $n$  with  $m < n$ , find  $m + n$ .
33. Let  $A_1, A_2, A_3, \dots, A_{12}$  be the vertices of a regular dodecagon. How many distinct squares in the plane of the dodecagon have at least two vertices in the set  $\{A_1, A_2, A_3, \dots, A_{12}\}$ ?
34. The solutions to the system of equations
- $$\begin{aligned} \log_{225} x + \log_{64} y &= 4 \\ \log_x 225 - \log_y 64 &= 1 \end{aligned}$$
- are  $(x_1, y_1)$  and  $(x_2, y_2)$ . Find  $\log_{30}(x_1 y_1 x_2 y_2)$ .

35. The Binomial Expansion is valid for exponents that are not integers. That is, for all real numbers  $x$ ,  $y$ , and  $r$  with  $|x| > |y|$ ,

$$(x + y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \dots$$

What are the first three digits to the right of the decimal point in the decimal

36. Harold, Tanya, and Ulysses paint a very long picket fence.
- - Harold starts with the first picket and paints every  $h^{\text{th}}$  picket;
  - Tanya starts with the second picket and paints every  $t^{\text{th}}$  picket; and
  - Ulysses starts with the third picket and paints every  $u^{\text{th}}$  picket.

Call the positive integer  $100h + 10t + u$  *paintable* when the triple  $(h, t, u)$  of positive integers results in every picket being painted exactly once. Find the sum of all the paintable integers.

37. Let  $ABCD$  and  $BCFG$  be two faces of a cube with  $AB = 12$ . A beam of light emanates from vertex  $A$  and reflects off face  $BCFG$  at point  $P$ , which is 7 units from  $\overline{BG}$  and 5 units from  $\overline{BC}$ . The beam continues to be reflected off the faces of the cube. The length of the light path from the time it leaves point  $A$  until it next reaches a vertex of the cube is given by  $m\sqrt{n}$ , where  $m$  and  $n$  are integers and  $n$  is not divisible by the square of any prime. Find  $m + n$ .
38. Let  $F(z) = \frac{z+i}{z-i}$  for all complex numbers  $z \neq i$ , and let  $z_n = F(z_{n-1})$  for all positive integers  $n$ . Given that  $z_0 = \frac{1}{137} + i$  and  $z_{2002} = a + bi$ , where  $a$  and  $b$  are real numbers, find  $a + b$ .
39. In triangle  $ABC$ , the medians  $\overline{AD}$  and  $\overline{CE}$  have lengths 18 and 27, respectively, and  $AB = 24$ . Extend  $\overline{CE}$  to intersect the circumcircle of  $ABC$  at  $F$ . The area of triangle  $AFB$  is  $m\sqrt{n}$ , where  $m$  and  $n$  are positive integers and  $n$  is not divisible by the square of any prime. Find  $m + n$ .

40. A set  $\mathcal{S}$  of distinct positive integers has the following property: for every integer  $x$  in  $\mathcal{S}$ , the arithmetic mean of the set of values obtained by deleting  $x$  from  $\mathcal{S}$  is an integer. Given that 1 belongs to  $\mathcal{S}$  and that 2002 is the largest element of  $\mathcal{S}$ , what is the greatest number of elements that  $\mathcal{S}$  can have?
41. Polyhedron  $ABCDEFGH$  has six faces. Face  $ABCD$  is a square with  $AB = 12$ ; face  $ABFG$  is a trapezoid with  $\overline{AB}$  parallel to  $\overline{GF}$ ,  $BF = AG = 8$ , and  $GF = 6$ ; and face  $CDE$  has  $CE = DE = 14$ . The other three faces are  $ADEG$ ,  $BCEF$ , and  $EFG$ . The distance from  $E$  to face  $ABCD$  is 12. Given that  $EG^2 = p - q\sqrt{r}$ , where  $p$ ,  $q$ , and  $r$  are positive integers and  $r$  is not divisible by the square of any prime, find  $p + q + r$ .