Euler, w4y2, Bulgarian Solitaire

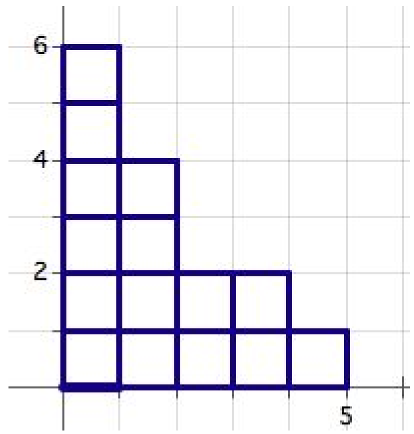
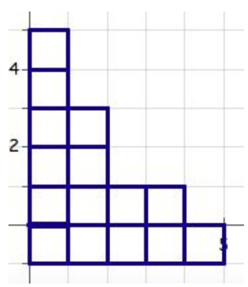
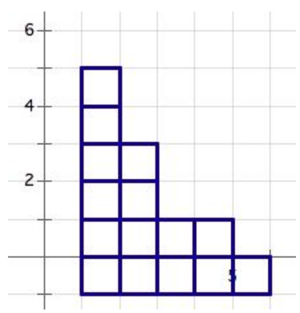
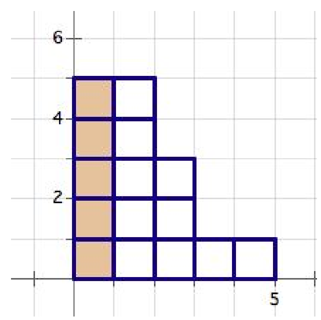
Supplies	Notes
M&M's (large bag + small paper cups, e.g. dixie cups)	Author: PJ Karafiol Email address: pjkarafiol@gmail.com Dates taught: 3/11, 3/18 Resources: The Bulgarian Solitaire and the Mathematics Around It Brandt, *Cycles of Partitions* Akin & Davis, *Bulgarian Solitaire* Igusa, *Solution of the Bulgarian Solitaire Conjecture*

Transferable Heuristics	• Change representations • Look for a quantity that is nonincreasing or nondecreasing
Objectives:	• Students can identify fixed points and n-cycles of iterative processes
Teacher Overview	

Narrative

Time/Description	Activity (include links to any handouts)	Teacher Notes
1:00-1:10	Each pair of students gets 15 M&M's in a cup. They divide the M&M's into several piles and then follow this rule: on each turn, take one M&M from each pile, and form a new pile with the gathered M&M's.	Eventually all configurations result in the collection (5,4,3,2,1). Questions to ask: - Is there a good way to represent this situation?

	What happens?	- Does it matter where you start out? - Are there simpler cases you can try?
1:15-1:25	Bring group together and develop some notation and terminology: - Represent each pile *P* with a number *k* of M&M's in that pile. - Represent the entire collection of piles with a nonincreasing sequence of integers that sum to *n,* the total number of M&M's. - Call the transformation that maps one collection to another via harvesting *H*. Then H^2 (C) = *H* (*H* (*C*)), etc. - A <u>fixed point</u> of *H* is a collection whose relative sizes are unchanged. So our claim is that *C** = (5,4,3,2,1) is a fixed point of *H*, and that for any other collection *C* of size 15, there exists an *n* such that H^n *(C) = C** What are some ways we can attack this problem? What are some weaker conjectures we can make?	Help students develop the notation: - What do we care about in each pile? - Does the order of the piles matter? (No.) Is there a convenient way to represent the situations so that equivalent collections look the same? - If we have a function H and a value x such that H(x) = x, what is x called? Students may suggest exhaustion, but it's probably helpful to get them to focus on smaller cases. Students may suggest that *H* is always periodic for any *C*, because the space of possible *C*'s is finite for any given *n*. Don't force this proof now.
1:25-1:45	Students explore cases where *n* < 15 in pairs or small groups.	Do all collections of size *n* fall into a single fixed point? Encourage students to draw flowcharts showing which configurations lead to which other configurations
1:45-1:55	Students present conjectures and proofs. These might include the following (only the first is important)	Students may be able to prove the main theorem for *n* = 6, 10 by exhaustion using flowcharts. At this point, if students haven't

	- If n is triangular, then all collections are eventually harvested to $(n, n-1, \ldots, 2, 1)$ - If n is not triangular, then every collection falls into a cycle. In particular, if $T_k < n < T_{k+1}$, then every collection of size n falls into a cycle of length k.	articulated it, develop the theorem that *H* is always periodic.
1:55-2:05	Break	
2:05-2:15	Represent the situation geometrically as a rotated Young tableau:  Represents (6,4,2,2,1). Then we are looking for a way to represent the operation of taking one box from each column and adding a new column with the removed boxes:    Moved down= harvested Make room for new column	Get the students to see that shifting down is better than "skimming off the top": "How can we represent harvesting as an operation that affects the entire tableau, not just the top boxes"? Once you get the boxes moved down, it's easy to suggest "How do we make room for another column?"

2:15-2:30	Identify a quantity that is monovariant (changes if at all in only one direction) when harvesting occurs. - Sum of coordinates of upper-right corner of each square - Sum of coordinates of center of the square (i.e. center of mass) Whatever it is, call it the "magic Sum" of the tableau *Can anyone finish the proof?*	"What technique have we used when we want to demonstrate something about a repeated process?" [Find a quantity that is invariant or changes in only one direction.] "Going to the right, what does that do the coordinates of points on the square? What about going down? What quantity <u>doesn't</u> change when we do that?"
2:30-??	If nobody has finished the proof, "What happens when the bottom is rotated back to the upper-left side? How does that affect the tableau's magic sum? - What happens to the sum after we rotate? - What if the leftmost column isn't tallest? - If the sum doesn't change, what does that tell you about the leftmost column after harvesting and restacking? (That it <u>is</u> the tallest)	Suppose there are k squares below the axis, with upper right coordinates (2,0), (3,0),... (k + 1,0). Then the magic sum is $(k+1)(k+2)/2 - 1= (k^2 + 3k)/2$. When rotated, their upper right corners are at (1,1), (1,2), (1,3), … , (1,k) with magic sum $k + (1 + 2 + 3 + … + k) = (k^2 + 3k)/2$. However, if the leftmost column is not the tallest, then we can imagine the topmost squares of the next rightmost column all moving left 1, which <u>decreases</u> the magic sum because the x-coordinates of all those squares decrease.
2:40	So how can we finish the proof? - Take a configuration C of T_k M&M's that never leads to (k, k-1, … , 1) and that has minimal magic sum. The ideal (k, k-1, … , 1) tableau has upper right corners along the line $y = k + 1 - x$, i.e. $x + y = k + 1$. If C is not this tableau, then it has at least one square directly above this line, i.e. $x + y = k + 2$, and at least one "hole" along this line, i.e. $x + y = k + 1$. But then the square cycles with length $k + 2$ and the hole cycles with length $k + 1$, so eventually the cycles coincide and the square winds up on top of the hole.	What does it mean to have a minimal magic sum? What does that tell you is happening whenever we harvest? (So whenever we harvest, the leftmost column is already tallest; all squares simply move down and to the right, until they reappear along the y-axis.) How long does it take to make a cycle? (The length of the diagonal, of course.)

Suggested Extension Questions

- The proof towards the end might involve too much lifting, so I'd encourage groups to generate more questions, for example:

- Does every configuration wind up in a single cycle? (Does the graph of the game separate out into distinct connected components?)
- How long is(are) the cycles for a given starting number of M&M's?
- Do the numbers in the configurations that are part of a cycle have any special patterns or properties?
- Are there certain configurations that can't be reached "late in the game"? How can you tell?

Teacher Reflections

- Lots of good stuff in Payton Euler. As expected, the hands-on start gave everyone access, and we periodically returned to test conjectures and try out different scenarios, which was important. It turned out that having students practice the notation by working through an example was pretty important.
- Students automatically started referring to configurations by ordered n-tuples, so all we had to do as a group was clean it up by specifying largest to smallest.
- I gave students graph paper, and students automatically started making Young tableaux, so again all we had to do was clean up by specifying "largest to smallest" and putting it on a coordinate plane. With nudging "How can we describe what happens to the entire configuration?" and "where will we put the new column?" students came up with the graphical representation.
- The question "What do we do when we have a repeated process -- what are some techniques we've used?" turned out to be very powerful. Students came quickly to "Find a quantity that is preserved or changes in only one direction," and from there, they generated "add the coordinates of a point to get a number representing each square on the diagram" and "add the numbers of all the squares to get a magic number for the diagram."
- Took full two hours but very worthwhile with many questions remaining for further investigation. Students discovered empirically that if $T_{k-1} < n < T_k$, then the cycle has k elements.
- In our lesson, we came up with our own name for the game. Students suggested "Willpower" (because it takes willpower not to eat the M&M's while you're playing) and then we changed it to "Millpomer" (m's pronounced like W's). We also called configurations "Mm-Piles".