

# Math Wrangle Solutions

## American Mathematics Competitions

August 4, 2012

1. The fact that  $OA = 1$  implies that  $BA = \tan \theta$  and  $BO = \sec \theta$ . Since  $\overline{BC}$  bisects  $\angle ABP$ , it follows that  $\frac{OB}{BA} = \frac{OC}{CA}$ , which implies  $\frac{OB}{OB+BA} = \frac{OC}{OC+CA} = OC$ . Substituting yields

$$OC = \frac{\sec \theta}{\sec \theta + \tan \theta} = \frac{1}{1 + \sin \theta}.$$

**OR**

Let  $\alpha = \angle CBO = \angle ABC$ . Using the *Law of Sines* on triangle  $BCO$  yields  $\frac{\sin \theta}{BC} = \frac{\sin \alpha}{OC}$ , so  $OC = \frac{BC \sin \alpha}{\sin \theta}$ . In right triangle  $ABC$ ,  $\sin \alpha = \frac{1-OC}{BC}$ . Hence  $OC = \frac{1-OC}{\sin \theta}$ . Solving this for  $OC$  yields

$$OC = \frac{1}{1 + \sin \theta}.$$

If circular arcs  $AC$  and  $BC$  have centers at  $B$  and  $A$ , respectively, then there exists a circle tangent to both  $\widehat{AC}$  and  $\widehat{BC}$ , and to  $\overline{AB}$ . If the length of  $\widehat{BC}$  is 12, then find the circumference of the smaller circle.

This problem was adapted from 2000 AMC 12, Problem 17.

By Heron's Formula the area of triangle  $ABC$  is  $\sqrt{(21)(8)(7)(6)}$ , which is 84, so the altitude from vertex  $A$  is  $2(84)/14 = 12$ . The midpoint  $D$  divides  $\overline{BC}$  into two segments of length 7, and the bisector of angle  $BAC$  divides  $\overline{BC}$  into segments of length  $14(13/28) = 6.5$  and  $14(15/28) = 7.5$  (since the angle bisector divides the opposite side into lengths proportional to the remaining two sides). Thus the triangle  $ADE$  has base  $DE = 7 - 6.5 = 0.5$  and altitude 12, so its area is 3.

This problem is adapted from 2000 AMC 12 Problem 19

Construct the circle with center  $A$  and radius  $AB$ . Let  $F$  be the point of tangency of the two circles. Draw  $\overline{AF}$ , and let  $E$  be the point of intersection of  $\overline{AF}$  and the given circle. By the *Power of a Point Theorem*,  $AD^2 = AF \cdot AE$  (see Note below). Let  $r$  be the radius of the smaller circle. Since  $\overline{AF}$  and  $\overline{AB}$  are radii of the larger circle,  $AF = AB$  and  $AE = AF - EF = AB - 2r$ . Because  $AD = AB/2$ , substitution into the first equation yields

$$(AB/2)^2 = AB \cdot (AB - 2r),$$

or, equivalently,  $\frac{r}{AB} = \frac{3}{8}$ . Points  $A$ ,  $B$ , and  $C$  are equidistant from each other, so  $\widehat{BC} = 60^\circ$  and thus the circumference of the larger circle is  $6 \cdot (\text{length of } \widehat{BC}) = 6 \cdot 12$ . Let  $c$  be the circumference of the smaller circle. Since the circumferences of the two circles are in the same ratio as their radii,  $\frac{c}{72} = \frac{r}{AB} = \frac{3}{8}$ . Therefore  $c = \frac{3}{8}(72) = 27$ .

**Note.** From any exterior point  $P$ , a secant  $PAB$  and a tangent  $PT$  are drawn. Consider triangles  $PAT$  and  $PTB$ . They have a common angle  $P$ . Since angles  $ATP$  and  $PBT$  intercept the same arc  $\widehat{AT}$ , they are congruent. Therefore triangles  $PAT$  and  $PTB$  are similar, and it follows that  $PA/PT = PT/PB$  and  $PA \cdot PB = PT^2$ . The number  $PT^2$  is called *the power of the point  $P$*  with respect to the circle. Intersecting secants, tangents, and chords, paired in any manner create various cases of this theorem, which is sometimes called *Crossed Chords*.

This problem is adapted from 2000 AMC 12 Problem 24

First we observe that, perhaps after a nonrepeating initial segment, the sequence  $f_1(11), f_2(11), \dots$  is periodic. To see this, it suffices to note that, for  $k < 1000$ ,

$$f_1(k) \leq f_1(999) = (9+9+9)^2 = 729 < 1000.$$

Next we compute  $f_n(11)$  for the first few values of  $n$ , in the hope that the length of the period is short. This expectation is reasonable since the terms of the sequence are perfect squares, and since there are only 31 perfect squares less than 1000. We find that

$$\begin{aligned} f_1(11) &= (1+1)^2 = 4, \\ f_2(11) &= f_1(f_1(11)) = f_1(4) = 4^2 = 16, \\ f_3(11) &= f_1(f_2(11)) = f_1(16) = (1+6)^2 = 49, \\ f_4(11) &= f_1(f_3(11)) = f_1(49) = (4+9)^2 = 169, \\ f_5(11) &= f_1(f_4(11)) = f_1(169) = (1+6+9)^2 = 256, \\ f_6(11) &= f_1(f_5(11)) = f_1(256) = (2+5+6)^2 = 169. \end{aligned}$$

We stop at this point, since  $f_n(11)$  depends only on  $f_{n-1}(11)$ , and hence the numbers 256 and 169 will continue to alternate. More precisely, for  $n \geq 4$ ,

$$f_n(11) = \begin{cases} 169, & \text{if } n \text{ is even,} \\ 256, & \text{if } n \text{ is odd.} \end{cases}$$

Since 1988 is even, it follows that  $f_{1988}(11) = 169$ .

This problem appeared as Problem 2 on the 1988 AIME.

Since

$$\log_8 x = \frac{1}{\log_x 8} = \frac{1}{3 \log_x 2} = \frac{1}{3} \log_2 x$$

and

$$\log_8(\log_2 x) = \frac{1}{3} \log_2(\log_2 x),$$

the given equation is equivalent to

$$\log_2(y/3) = (1/3) \log_2 y, \tag{1}$$

where  $y = \log_2 x$ . From (1),  $\log_2(y/3)^3 = \log_2 y$ , hence  $(y/3)^3 = y$ ,  
i.e.,

$$y(y^2 - 27) = 0. \tag{2}$$

Since  $y \neq 0$  (for otherwise, neither side of (1) would be defined), it follows from (2) that  $y^2 = (\log_2 x)^2 = 27$ .

This problem appeared as Problem 3 on the 1988 AIME.

If  $n$  is a positive integer, then

$$\sum_{k=1}^n |x_k| - \left| \sum_{k=1}^n x_k \right| \leq \sum_{k=1}^n |x_k| < n, \quad (1)$$

since  $\left| \sum_{k=1}^n x_k \right| \geq 0$  and  $|x_k| < 1$  for  $1 \leq k \leq n$ . Since it is given that

$$\sum_{k=1}^n |x_k| - \left| \sum_{k=1}^n x_k \right| = 19, \quad (2)$$

it follows that  $19 < n$ . Thus the answer is 20 if there is a solution to (2) with  $n = 20$ . One such solution is

$$x_k = \begin{cases} .95 & \text{if } k \text{ is odd,} \\ -.95 & \text{if } k \text{ is even.} \end{cases}$$

This problem appeared as Problem 4 on the 1988 AIME.

The divisors of  $10^{99}$  are of the form  $2^a \cdot 5^b$ , where  $a$  and  $b$  are integers with  $0 \leq a \leq 99$  and  $0 \leq b \leq 99$ . Since there are 100 choices for both  $a$  and  $b$ ,  $10^{99}$  has  $100 \cdot 100$  positive integer divisors. Of these, the multiples of  $10^{88} = 2^{88} \cdot 5^{88}$  must satisfy the inequalities  $88 \leq a \leq 99$  and  $88 \leq b \leq 99$ . Thus there are 12 choices for both  $a$  and  $b$ ; i.e.,  $12 \cdot 12$  of the  $100 \cdot 100$  divisors of  $10^{99}$  are multiples of  $10^{88}$ . Consequently, the desired probability is  $\frac{m}{n} = \frac{12 \cdot 12}{100 \cdot 100} = \frac{9}{625}$ .

This problem appeared as Problem 5 on the 1988 AIME.

Call a real-valued function  $f$  *very convex* if

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x-y|$$

holds for all real numbers  $x$  and  $y$ . Prove that no very convex function exists.

**First Solution.** Fix  $n \geq 1$ . For each integer  $i$ , define

$$\Delta_i = f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right).$$

The given inequality with  $x = (i + 2)/n$  and  $y = i/n$  implies

$$\frac{f\left(\frac{i+2}{n}\right) + f\left(\frac{i}{n}\right)}{2} \geq f\left(\frac{i+1}{n}\right) + \frac{2}{n},$$

from which,

$$f\left(\frac{i+2}{n}\right) - f\left(\frac{i+1}{n}\right) \geq f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right) + \frac{4}{n},$$

that is,  $\Delta_{i+1} \geq \Delta_i + 4/n$ . Combining this for  $n$  consecutive values of  $i$  gives  $\Delta_{i+n} \geq \Delta_i + 4$ . Summing this inequality for  $i = 0$  to  $i = n - 1$  and canceling terms yields

$$f(2) - f(1) \geq f(1) - f(0) + 4n.$$

This cannot hold for all  $n \geq 1$ . Hence there are no very convex functions.

**Second Solution.** We show by induction that the given inequality implies

$$\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \geq 2^n |x - y|$$

for all nonnegative integer  $n$ . This will yield a contradiction, because for fixed  $x$  and  $y$  the right side gets arbitrarily large, while the left side remains fixed.

We are given the base case  $n = 0$ . Now if the inequality holds for a given  $n$ , then for  $a, b$  real,

$$\begin{aligned} \frac{f(a) + f(a+2b)}{2} &\geq f(a+b) + 2^{n+1}|b|, \\ f(a+b) + f(a+3b) &\geq 2(f(a+2b) + 2^{n+1}|b|), \end{aligned}$$

and

$$\frac{f(a+2b) + f(a+4b)}{2} \geq f(a+3b) + 2^{n+1}|b|.$$

Adding these three inequalities and canceling terms yields

$$\frac{f(a) + f(a+4b)}{2} \geq f(a+2b) + 2^{n+3}|b|.$$

Setting  $x = a$ ,  $y = a + 4b$ , we obtain

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + 2^{n+1}|x-y|,$$

and the induction is complete.

**Third Solution.** Rewrite the condition as

$$f(x) - f\left(\frac{x+y}{2}\right) \geq f\left(\frac{x+y}{2}\right) - f(y) + 2|x-y|.$$

For any positive integer  $n$ ,

$$\begin{aligned} f(1) - f(0) &= f(1) - f(1/2) + f(1/2) - f(0) \\ &\geq f(1/2) - f(0) + 2 + f(1/2) - f(0) = 2[f(1/2) - f(0)] + 2 \\ &= 2[f(1/2) - f(1/4) + f(1/4) - f(0)] + 2 \\ &\geq 2[f(1/4) - f(0) + 1 + f(1/4) - f(0)] + 2 \\ &= 4[f(1/4) - f(0)] + 4 \\ &= \dots \geq 2^n[f(1/2^n) - f(0)] + 2n. \end{aligned}$$

Similarly,  $f(-1) - f(0) \geq 2^n[f(-1/2^n) - f(0)] + 2n$ . But

$$f(1/2^n) + f(-1/2^n) \geq 2f(0) + 1/2^{n-2} > 2f(0).$$

Thus, for each  $n$ , at least one of  $f(1/2^n) - f(0)$  and  $f(-1/2^n) - f(0)$  is greater than 0. It follows that at least one of  $f(1) - f(0)$  and  $f(-1) - f(0)$  is greater than  $2n$  for all  $n \geq 1$ , which is impossible.

Hence there is no very convex function.

This problem appeared as Problem 1 on the 2000 USAMO.