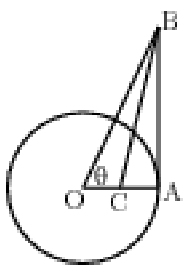
Math Wrangle Problems

American Mathematics Competitions

August 4, 2012

1. A circle centered at O has radius 1 and contains the point A. Segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on $\overline{OA}$ and $\overline{BC}$ bisects $\angle ABO$, then express OC as simply as possible in terms of trigonometric functions of θ.



2. In triangle ABC, $AB = 13$, $BC = 14$, and $AC = 15$. Let D denote the midpoint of $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle BAC. What is the area of the triangle ADE?

3. If circular arcs AC and BC have centers at B and A, respectively, then there exists a circle tangent to both $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$, and to $\overline{AB}$. If the length of $\overset{\frown}{BC}$ is 12, then find the circumference of the smaller circle.

4. For any positive integer k, let $f_1(k)$ denote the square of the sum of the digits of k. For $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1988}(11)$.

5. Find $(\log_2 x)^2$ if $\log_2(\log_8 x) = \log_8(\log_2 x)$.

6. Suppose that $|x_i| < 1$ for $i = 1, 2, \ldots, n$. Suppose further that

$$|x_1| + |x_2| + \cdots + |x_n| = 19 + |x_1 + x_2 + \ldots + x_n|.$$

What is the smallest possible value of n?

7. Let m/n, in lowest terms, be the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}. Find that probability m/n.

8. Call a real-valued function f *very convex* if

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x-y|$$

holds for all real numbers x and y. Prove that no very convex function exists.