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A Guide for the Perplexed: What Mathematicians Need to Know to Understand Philosophers of Mathematics

1. Introduction

When I received the invitation to read a paper on the philosophy of mathematics at this conference, my first thought was to read one of the papers that I'm working on for publication. But then I thought it might be better to give a talk that provides a sort of guide for mathematicians who don't know much about the philosophy of mathematics--a guide that explains how to read philosophers of mathematics. So that's what I'm going to do. My hope is to make clear for mathematicians what philosophers of mathematics are really up to and, also, to eliminate some confusions. I should say at the start, however, that the picture of the philosophy of mathematics that I'm going to provide here is controversial. Some philosophers of mathematics would agree with me, but others would disagree. This is par for the course in philosophy--we disagree about almost everything. I think that the picture that I'll paint here of the philosophy of mathematics is accurate, and I think there are compelling arguments for the claim that this picture is accurate. If I were writing a different paper--one that was intended more for philosophers rather than for mathematicians--I would give those arguments. But I'm not going to worry much about this here. I'm simply going to provide my description of the discipline, and my guide for how to read philosophers of mathematics, and I'm going to let it be controversial. I will try to indicate where it's controversial, but I won't say very much to justify my description.

2. Clearing Up Some Confusions About the Philosophy of Mathematics

The two main confusions about the philosophy of mathematics that I want to try to clear up have to do with the following questions:

> 1. What is the relationship between mathematics and the philosophy of mathematics? I think that a lot of mathematicians might think things like this: "Who are these outsiders telling us about our discipline?" I want to paint a picture of the relationship that might make you think, "Oh, I see; they could have something to tell us here."
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> 2. What is the core thing that philosophers of mathematics are doing? In other words, what kinds of theories are they putting forward?

These two questions are deeply related. I think there is a lot of confusion about question number 2, and this leads to confusion about question number 1. If we can get clear on question number 2, this will bring with it an answer to question number 1. So let me start with question number 2. What kinds of theories are philosophers of mathematics putting forward?

I think that a lot of people who don't work in the philosophy of mathematics (mathematicians included) think that what's mainly going on in the philosophy of mathematics is the following: (a) there's one group of people putting forward a pie-in-the-sky theory about weird, metaphysically occult objects (in particular, abstract mathematical objects like numbers) that exist in a non-physical, non-mental, non-spatiotemporal Platonic realm, and (b) there's another group of people arguing against this seemingly crazy view--e.g., by claiming that mathematics is really just a mental construction, or a social construction, or some such thing. But while there's definitely some of this going on in the philosophy of mathematics, this is not the main thing that's going on, and it is driven by something else, something that's more primary.

But let me back up a bit and introduce some lingo going. First, let's say that an abstract object, or a platonic object, is a non-physical, non-mental, non-spatiotemporal object. We can call the view that there are such objects platonism; and the view that there are no such things is anti-platonism. There are various kinds of objects that platonists think are abstract objects, but the only ones that will matter here are mathematical objects--i.e., things like numbers, sets, functions, and so on. Metaphorically, we can say that these objects exist in "platonic heaven"--which we can think of as sort of like the Christian heaven, except that what exists there aren't angels and Gods and ghosts, but numbers and sets and so on. But this is a metaphor; the official platonist view is that mathematical objects like numbers exist, and they're non-mental, non-physical, and non-spatiotemporal. (Platonism obviously goes back to Plato (see, e.g., the Meno and the Phaedo), but it has also been endorsed by numerous people since then, including Frege (1884), Russell (1912), and Gödel (1964).)

Second, I want to introduce the notion of an ontological theory. We can say that ontology is the branch of rational inquiry that's concerned with cataloguing the various kinds of objects that exist. Thus, a specific ontological theory is a theory about what sorts of things really exist. Thus, for instance, the claim that there are mermaids is a false ontological theory, and the claim that there are Tasmanian devils is a true ontological theory. Platonism, then, as I defined it above, is an ontological theory. It is the theory that abstract objects (i.e., non-physical, non-mental, non-spatiotemporal objects) exist.

OK, so given all of this, I can restate the point I made a few minutes ago. I think a lot of people think that the main thing that's going on in the philosophy of mathematics is ontology. In particular, it might seem that philosophers of mathematics are arguing about whether a certain ontological theory--namely, platonism--is true. But, again, I want to claim that this is at best an oversimplification. There is certainly some ontology going on in the philosophy of mathematics, and for a lot of philosophers of mathematics, this is the ultimate point of their work. But even among those who see ontology as the final point, this is not the main thing that's driving their arguments. In short, ontological theories like platonism and anti-platonism are best thought of as following from theories of a completely different kind, and it is the development of theories of this other kind that form the primary core of the philosophy of mathematics.

The other kind of theory that I'm talking about here is a semantic theory. A semantic theory is a theory about what certain expressions mean (or refer to) in a specific language. So, for instance, the claim that the term 'Mars' refers (in English) to the Empire State Building is a false semantic theory, and the claim that 'Mars' refers (in English) to the forth planet from the sun is a true semantic theory.

Here's an important point to note about semantic theories of the kind I'm talking about: if the language in question is a natural language--if it's a language that's actually spoken by real people--then semantic theories of that language will be empirical theories. For instance, it's an empirical fact about us speakers of English that we use 'Mars' in the way that we do. We might have used it to refer to the Empire State Building; we just in fact don't.

In any event, my claim here is that philosophers of mathematics are primarily concerned with constructing semantic theories. In particular, their aim is to develop a semantic theory for the language of ordinary mathematical discourse--or as philosophers sometimes call it, mathematese. I say this is what they're primarily concerned with. Many of them are ultimately interested in ontology, but for most these people, the arguments for their ontological theories are almost always motivated by the adoption of semantic theories. And some philosophers of mathematics don't care very much at all about ontology. But they almost all care about semantics. It's hard to think of a single example of a serious philosophy of mathematics that doesn't centrally involve a semantic theory of mathematical discourse. If you like, you can think of it like this: philosophers of mathematics are centrally concerned with developing level-headed empirical theories of the semantics of mathematical discourse; and then many of them use these theories to motivate bizarre ontological theories.

So we now have an answer to question number 2: the core thing that philosophers of mathematics are doing is constructing empirical theories of the semantics of ordinary mathematical discourse. So they are more like linguists than anything else.

And now, given this, we can answer question number 1, i.e., the question about the relationship between mathematicians and philosophers of mathematics. The relationship is analogous to the relationship between a native speaker of French and a certain sort of linguist—in particular, a grammarian of French whose native tongue is English but who has learned a good deal of French in order to construct a grammar for that language. There is an obvious sense in which the native speaker of French knows her language better--indeed, much better--than the linguist does. But the linguist has been trained to construct syntactic theories, and most native speakers of French have not. Thus, while the linguist has to respect the linguistic intuitions of native speakers, he cannot very well ask them what the right theory is. Likewise, while it is obvious that mathematicians know mathematics (and the language of mathematics) better than philosophers do--indeed, much better--most of them have not been trained to construct semantic theories in the way that philosophers have. So while philosophers of mathematics have to respect the intuitions of mathematicians, they cannot very well ask them what the right theory is.

So I hope this gives us a way of understanding why philosophers of mathematics might have something to offer here, i.e., why they're not just spouting off about something that they know nothing about.

(I said I would tell you what was controversial about my view. Well, let me pause here to point out that a lot of philosophers of mathematics would resist the idea that they are like linguists. For instance, a lot of them would say that they are concerned more with ontology than with semantics because their main goal is to determine the ultimate metaphysical nature of reality. I think that for the most part, these people are simply mistaken about what they're doing. I can't take the time to argue for this here, but if I were to do this, I would do it by analyzing the papers of these people and showing how--unbeknownst to them--their arguments are ultimately driven by semantic theories. It would take at least a whole essay to do this well, so I'm not going to get into this here. But let me make two points about this before moving on. First, as I'll point out below, what I say in the remainder of this essay can at least be seen as a sort of partial argument for my view. And second, I just want to point out that there is nothing very strange about the idea that people can be mistaken about what their own work is about. Indeed, anyone who endorses a view in the philosophy of mathematics is committed to saying that some mathematicians are mistaken about what their work is about. Suppose, for instance, that you and I endorsed a platonistic philosophy of mathematics. Well, insofar as lots of mathematicians explicitly reject platonism, we would be committed to saying that these mathematicians are mistaken about what their own mathematical work is about. In other words, platonists need to say that the work of mathematicians is about abstract objects even if some mathematicians don't realize this. Likewise, I am claiming here that the philosophy of mathematics is largely about semantics even if some philosophers of mathematics don't realize this.)

3. Doing Some Empirical Semantics

I now want to give you an example of what I've been talking about. I want to construct an entirely empirical argument for a specific theory of the semantics of mathematese, i.e., for the language of mathematics. (And I should note before I start that we can view what I'll be saying here as at least partially justifying my view of the philosophy of mathematics as largely about semantics. For what I'll be doing in what follows is taking a traditional philosophical argument and explaining how to read it as being largely about semantics.) In any event, the semantic theory that I will be arguing for can be put like this:

> Semantic Platonism: Ordinary mathematical sentences like '2 + 2 = 4' and '3 is prime' are straightforward claims about abstract objects (or at any rate, they purport to be about abstract objects).

Now, the first point I want to make about this theory is that it is not an ontological theory, and it doesn't imply any ontological theories. In particular, it doesn't imply that platonism is true; i.e., it doesn't imply that there actually exist any abstract objects. This is an extremely important point, and it's worth pausing to make sure the point is very clear. Let me do this by switching to a different example. Suppose that a team of Martian linguists landed on Earth and started trying to construct semantic theories for our languages. And suppose in particular that they happened upon a Christian community that kept employing the term 'God'. Next, suppose that one of the Martians proposed the hypothesis that they are using the term ‘God’ as a nickname for Gödel. And finally, suppose that another of the Martians disagreed with this theory and proposed the following alternative:

> Semantic Theism: The term ‘God’ refers (in English) to an omniscient, omnipotent, benevolent Being who created the world (or at any rate, the term ‘God’ purports to refer to such a Being).

The Martian who puts this theory forward might not himself believe in God. His theory is a theory about how Christians use a certain term. Thus, semantic theism does not imply theism; i.e., it doesn’t imply that God actually exists. And likewise, semantic platonism does not imply platonism; you can endorse this theory without believing in abstract objects. So semantic theism could be true even if theism is false. Indeed, I take it as more or less obvious that semantic theism is true; but this doesn’t tell us anything about whether theism is true, i.e., whether God exists. Likewise, semantic platonism could be true even if platonism is false.

OK, so that’s what semantic platonism says. It’s a straightforward empirical theory about the ordinary usage of mathematical sentences. And now I want to give you a straightforwardly empirical argument for this theory. The first premise of the argument is as follows:

> (1) Ordinary mathematical sentences like ‘2 + 2 = 4’ and ‘3 is prime’ should be interpreted at face value; i.e., they should be interpreted literally. For instance, ‘3 is prime’ should be interpreted as having the following logical form: Object O has property P; or as philosophers would put it, it has the form Fa. Thus, what ‘3 is prime’ says is that a certain object (namely, the number 3) has a certain property (namely, the property of being prime).

This premise is extremely plausible, but let me say a bit to explain it and justify it. Consider the following sentences:

> (M) Mars is round.
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> (O) Obama is a politician.
>
> (E) The Eiffel Tower is made of metal.

All three of these sentence have the same logical form; they all say that a certain object has a certain property. In other words, they all have the form: Object O has property P. Now, on the surface, it seems that ‘3 is prime’ has this form as well; it seems to say that a certain object (namely, 3) has a certain property (namely, primeness). But we have to be careful here. For, sometimes, when a sentence seems on the surface to have one logical form, it really has a different logical form. Here’s an example:

> (A) The average accountant has 2.4 children.

The surface form of this sentence is similar to the above sentences; on the surface, it seems to be saying that a certain object (namely, the average accountant) has a certain property (namely, the property of having 2.4 children). But, of course, this isn't really what this sentence says. The deep logical form of the sentence is as follows: On average, accountants have 2.4 children.

Now, given this, one might try to argue that while '3 is prime' seems on the surface to say that a certain object has a certain property, that's not the deep logical form of the sentence. But this would be wildly implausible. When we interpret people's speech, the default setting is always to take them to be speaking literally, or at face value. To motivate a non-face-value interpretation for a given sentence, we have to motivate the claim that the speaker or speakers in question have a positive intention to be saying something other than what the sentence says literally. And there has to be empirical evidence for this claim. In the case of (A), there is a mountain of evidence for the claim that when ordinary people utter sentences like this, they don't mean to be saying what the sentence says on the surface--they actively intend to be saying something else--and so they should not be interpreted as speaking literally. But in the case of ordinary mathematical sentences like '3 is prime', there is no evidence for the claim that people mean to be speaking non-literally, or metaphorically, and so we should interpret them as speaking literally. In other words, when people utter '3 is prime', we should interpret them as saying exactly what they seem to be saying--namely, that a certain number (namely, 3) has the property of being prime.

So that is the argument for premise (1). Again, this is an empirical argument for an empirical hypothesis about ordinary speakers of mathematical language--the claim is simply that these people are speaking literally when they say things like '3 is prime' and 'There are infinitely many primes'. The second premise in the argument can be put like this:

> (2) Given that ordinary mathematical sentences like '2 + 2 = 4' and '3 is prime' should be interpreted at face value--i.e., as making straightforward claims about certain objects (namely, numbers)--we can't interpret them as being about physical or mental objects, and so we have to interpret them as being about abstract objects, i.e., non-physical, non-mental, non-spatiotemporal platonic objects (or more precisely, we have to interpret them as purporting to be about abstract objects).

Now, let me remind you of two points I've already made: first, premise (2) should not be taken as implying that platonism is true (i.e., that there really are abstract objects); and second, premise (2) is an empirical claim. The idea here is that this is the best way to interpret the ordinary mathematical assertions of ordinary people and ordinary mathematicians.

Note, however, that the advocate of (2) does not have to say that ordinary people consciously intend to be talking about abstract objects; the claim is going to be that the only view that's not inconsistent with the linguistic intentions of ordinary speakers is the platonistic interpretation. But let me start at the beginning.

The first point to note in connection with premise (2) is that the only options for what numerals like '3' might refer to (or purport to refer to) are physical objects, mental objects, and abstract objects. If an object O is a real thing, then it is either an ordinary physical object existing in the physical world; or a mental object, e.g., an idea in one of our heads (of course, if you're a materialist about the mind, then you'll want to say that mental objects are just a special kind of physical object, but let's not worry about whether this is true); or an abstract object. There just don't seem to be any other options.

(I suppose you might think that numbers are <u>social</u> objects, but social objects are higher-level objects that have to ultimately reduce to physical objects, mental objects, or abstract objects. Consider, e.g., the convention to stop at red lights. This isn't a free-floating social object. If it's not an abstract object of some kind, then it presumably reduces to individual mental objects--e.g., my belief that we're supposed to stop at red lights, and my intention to follow this rule, and so on. All it <u>is</u> for there to be a social convention of the above kind is for a sufficiently large percentage of people to have beliefs and intentions like this. So, again, it seems to me that social objects ultimately reduce to physical objects, mental objects, and abstract objects, and so it seems that the only <u>bottom-level</u> objects that we should believe in are physical objects, mental objects, and abstract objects.)

Thus, if we can give empirical reasons for thinking that mathematical terms like '3' cannot be interpreted as referring (or purporting to refer) to physical or mental objects, then (assuming that we don't also have reasons for thinking that they can't be taken as referring (or purporting to refer) to abstract objects), we will have good reason to adopt the semantic platonist view that we ought to interpret these terms as referring (or purporting to refer) to abstract objects. I'll say more about this below, but for now, let's proceed with the empirical reasons for rejecting physicalistic and psychologistic semantic theories. And let me begin by putting these two theories on the table for discussion:

> <u>Semantic Physicalism</u>: Ordinary mathematical sentences like '2 + 2 = 4' and '3 is prime' are best interpreted as straightforward claims about ordinary physical objects. (One famous advocate of a view like this is John Stuart Mill (1843).)
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> <u>Semantic Psychologism</u>: Ordinary mathematical sentences like '2 + 2 = 4' and '3 is prime' are best interpreted as straightforward claims about ordinary mental objects--i.e., things like <u>ideas</u> that actually exist inside of our heads. (Advocates of views of this kind include Husserl (1891), Brouwer (1912 and 1948), and Heyting (1956).)

I want to provide some reasons for thinking that that these two theories are simply unacceptable. Let me start with semantic physicalism.

One problem with semantic physicalism is that if it were right, then it would be reasonable to worry that there just aren't enough objects in the world to make our mathematical theories true. To appreciate this point, imagine a mathematics professor teaching Euclid's proof of the claim that there are infinitely many prime numbers, and imagine a student raising her hand with the following objection: "There couldn't be infinitely many prime numbers because my physics professor told me that there are only finitely many physical objects in the whole universe."

Or to make the problem even more vivid, imagine that after being taught Cantor's theorem (that there are infinitely many transfinite cardinals that keep getting bigger and bigger without end), a student said this: "There couldn't be infinitely many transfinite cardinals that get bigger and bigger without end because my physics professor ensures me that there just aren't that many physical objects in the universe."

It seems reasonable to think that these two students just don't understand; they don't understand what the two proofs are supposed to show. In the context of Euclid's and Cantor's proofs, it just doesn't matter how many physical objects there are. Even if it's true that there are only finitely many physical objects in the universe, this is simply not a good reason to reject the two proofs. And the only reasonable conclusion we can draw from this, I think, is that the two theorems--i.e., the sentences that say that there are infinitely many primes and infinitely many transfinite cardinals--should not be interpreted as being about physical objects.

A second problem with semantic physicalism is that it implies that our mathematical theories are empirical theories and that the right methodology for determining whether there are, say, infinitely many primes would involve an empirical investigation into the number of physical objects in the universe. In other words, semantic physicalism implies that the proofs of Euclid and Cantor are not just mistaken but completely wrongheaded in their methodology. If semantic physicalism were true, then Euclid and Cantor should have used empirical methods. But, of course, this is crazy. There's nothing wrong with the methodology of mathematical proof; the problem here is that semantic physicalism is simply false.

Very quickly, here's a third argument against semantic physicalism: when we apply this semantic theory to set theory, we get the conclusion that expressions that are supposed to refer to set are supposed to refer to piles of physical stuff. But this can't be right, because corresponding to every pile of physical stuff—indeed, every individual physical object—there are infinitely many sets. Corresponding to a ball, for instance, is the set containing the ball, the set containing its molecules, the set containing that set, and so on. Clearly, these sets are not supposed to be purely physical objects, because (a) they are all supposed to be distinct from one another, and (b) they all share the same physical base (i.e., they're all made of the same matter and have the same spatiotemporal location). Thus, there must be something non-physical about these sets, over and above the physical base that they all share. So sets cannot be purely physical objects.

Before we move on, it's worth reminding ourselves that the above arguments are empirical arguments, and the conclusion is an empirical claim about what we mean when we use mathematical language. What these arguments show is that there is simply no plausible way to interpret ordinary mathematical claims as being claims about physical objects. For, in short, facts about how many physical objects there are in the universe are completely irrelevant to ordinary claims about how many mathematical objects there are.

Let's move on now to semantic psychologism--i.e., to the view that ordinary mathematical sentences are supposed to be claims about mental objects like ideas in our heads; e.g., an advocate of this view might say that '3 is prime' says that a certain mental object (namely, the idea of 3) has the property of being prime. We can construct arguments against this view that are very similar to the arguments that I just ran against semantic physicalism.

The first problem with semantic psychologism is that if it were right, then it would be reasonable to worry that there just aren't enough mental objects in the world to make our mathematical theories true. To appreciate this, imagine that after being taught Euclid's proof or Cantor's proof, a student raised his hand and said: "There couldn't be infinitely many prime numbers (or infinitely many transfinite cardinals) because my psychology professor told me that there are only finitely many ideas in each human head, and my astronomy professor told me there are no aliens with thoughts, and so there are only finitely many mental objects in the whole universe." It seems reasonable to think that this student just doesn't understand. In the context of Euclid's and Cantor's proofs, it just doesn't matter how many mental objects there are in the universe. Even if it's true that there are only finitely many mental objects in the universe, this is simply not a good reason to reject the two proofs. And the only reasonable conclusion we can draw from this, I think, is that the two theorems--i.e., the sentences that say that there are infinitely many primes and infinitely many transfinite cardinals--should not be interpreted as being about actual mental objects that exist in our heads.

It's important to note that the worry here is not that humans can't conceive of an infinite set. The worry has to do with the number of actual mental objects (i.e., distinct number-ideas) that are actually residing in human heads. Semantic psychologism implies that in order for standard arithmetical theories like Peano Arithmetic (PA) to be true, there has to be an infinite number of these actual mental objects. Why? Because PA implies that there are infinitely many numbers; it implies that there is such a thing as the number 1, and there is such a thing as the number 2, and 2 is not identical to 1, and so on. Thus, if semantic psychologism were right, then the truth of PA would depend on their actually existing infinitely many distinct number-ideas in human heads. But, in fact, the truth of PA clearly doesn't depend on this; if you're worried that PA might be false because there aren't enough actual ideas to go around, then that just shows that you don't understand what PA says. And so the conclusion we should draw here is that semantic psychologism is false.

A second problem with semantic psychologism is that it implies that our mathematical theories are empirical theories and that the right methodology for determining whether there are, say, infinitely many primes would involve an empirical investigation into the number of actual number-ideas that exist in the universe. In other words, semantic psychologism implies that the proofs of Euclid and Cantor are not just mistaken but completely wrongheaded in their methodology. If semantic psychologism were true, then Euclid and Cantor should both have used empirical methods. But, of course, this is crazy. There's nothing wrong with the method of mathematical proof; the problem here is that semantic psychologism is false.

Before moving on, I want to make a few points about all of this. First, it might seem that these arguments are directed at a silly or trivial version of semantic psychologism that no one would ever endorse. Well, I agree with that. The view is crazy. (As Frege says (1884, section 27), "Weird and wonderful...are the results of taking seriously the suggestion that number is an idea.") But the problem is that there is no way to get rid of the craziness, or the silliness, without altering the view in a way that makes it the case that it's no longer a psychologistic view at all. Suppose, for instance, that someone said something like this:

> The psychologistic view isn't that mathematics is about actual ideas that really exist inside of human heads. We can take the view to be about what it's possible to do in our heads. For instance, to say that there are infinitely many prime numbers is not to say that there really exists an actual infinity of prime-number ideas inside of human heads; it's to say that it's possible to construct infinitely many prime numbers in our heads.

There are a few problems with this view. In the present context, the main problem is that the view here isn't a psychologistic view at all, and so it's no defense against the above objection. Semantic psychologism is the view that mathematical claims are about mental objects. The above view rejects this, and so it's not a version of semantic psychologism. Rather, it's a version of non-literalism; in other words, it rejects the above thesis that when we say things like '3 is prime', we're speaking literally; on the view in question, '3 is prime' doesn't really say that a certain object (namely, 3) is prime; rather, it says something about what it's possible for humans to do. But as an empirical hypothesis about what ordinary people actually mean when they utter sentences like '3 is prime', this is just really implausible; there's simply no evidence that people really mean to say things like this when they utter sentences like '2 + 2 = 4' and '3 is prime'. (The only people who ever mean things like this by mathematical claims are people who are worried about philosophy.)

In any event, if we stick with a genuinely psychologistic semantics, the view is crazy, and the above arguments show that. And it's important to remember that the claim here is entirely semantic. None of this is to deny the ontological thesis that there are number-ideas in our heads. I take it that this is entirely obvious. What the above arguments show is that numerals like '3' shouldn't be taken to refer to these ideas.

Similarly, it should also be clear that studies that aim to show that our mathematical ideas originate in our brains (I'm thinking here of the work of people like Stanislas Dehaene) are completely irrelevant to a defense of semantic psychologism. It may be true that our mathematical ideas originate in our brains, and that platonic heaven didn't need to exist in order for us to come up with all of the mathematics that we have come up with; but it just doesn't follow from this that numerals like '3' are supposed to refer to things inside our heads. An analogy here is the God case; you might think that our God thoughts originate in our brains, and that God didn't need to exist in order for us to come up with these thoughts; but it simply doesn't follow that the term 'God' is supposed to refer to something inside our heads, and in fact, it is entirely obvious that it's not supposed to refer to something inside our heads; it's supposed to refer to a creator of the world (you should admit that this is true whether you believe in the existence of such a creator of not).

Finally, it's worth noting that the above arguments should not be taken as arguments against intuitionism. It is often thought that intuitionism is a form of psychologism, but this is a mistake. What's true is that many intuitionists--most notably, Brouwer (1912 and 1948), and Heyting (1956)--have also endorsed psychologism. But intuitionism is perfectly consistent with platonism and other anti-platonistic views, and psychologism is consistent with a rejection of intuitionism.

In any event, we now have arguments against semantic physicalism and semantic psychologism, and if we combine these arguments with the above argument for premise (1), we get an argument for semantic platonism, i.e., for the claim that sentences like '3 is prime' are best interpreted as being about abstract objects (or at least purporting to be about abstract objects). The argument goes like this:

> (1) Ordinary mathematical sentences like '2 + 2 = 4' and '3 is prime' should be interpreted at face value. For instance, '3 is prime' should be interpreted as having the following logical form: Object O has property P; or as philosophers would put it, it should be interpreted as having the form Fa. Thus, what '3 is prime' says is that a certain object (namely, the number 3) has a certain property (namely, the property of being prime).
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> (2) Given that ordinary mathematical sentences like '2 + 2 = 4' and '3 is prime' should be interpreted at face value--i.e., as making straightforward claims about certain objects (namely, numbers)--we can't interpret them as being about physical or mental objects, and so we have to interpret them as being about abstract objects, i.e., non-physical, non-mental, non-spatiotemporal platonic objects (or more precisely, we have to interpret them as purporting to be about abstract objects). Therefore,
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> (3) Semantic platonism is true. In other words, ordinary mathematical sentences like '2 + 2 = 4' and '3 is prime' are straightforward claims about abstract objects (or at any rate, they purport to be about abstract objects).

Now, you might object here that just as there are reasons to resist semantic physicalism and semantic psychologism, so too there are reasons to resist semantic platonism. For you might think it's implausible that ordinary people intend to be speaking of abstract objects when they say things like '3 is prime'. But semantic platonists don't need to say that people have such intentions, and indeed, they shouldn't say this. What they should say is that (a) people are best interpreted as speaking literally when they say things like '3 is prime', and so these sentences have to be taken as being about objects (in particular, numbers); and (b) our semantic intentions are incompatible with semantic physicalism and semantic psychologism, and so there is no way to interprete us as talking about physical or mental objects when we say things like '3 is prime' (this is what the above arguments show); and (c) there's nothing in our intentions that's incompatible with semantic platonism; and so (d) even if people don't have a positive intention to be referring to abstract objects when they say things like '3 is prime', the best interpretation of these utterances has it that they are about abstract objects (or at least that they purport to be about such objects).

So we're done--we have a purely empirical argument for semantic platonism.

4. From Level-Headed Empirical Semantics to Crazy Ontology

OK, so that's the argument for semantic platonism. It is an entirely empirical argument, and it is extremely compelling. But we can now use this argument to argue for the ontological thesis that platonism is true. The argument goes like this:

> (i) Semantic platonism is true--i.e., ordinary mathematical sentences like '2 + 2 = 4' and '3 is prime' are straightforward claims about abstract objects (or at any rate, they purport to be about abstract objects). Therefore,
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> (ii) Mathematical sentences like '2 + 2 = 4' and '3 is prime' could be true only if platonism were true--i.e., only if abstract objects existed. But
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> (iii) Mathematical sentences like '2 + 2 = 4' and '3 is prime' are true. Therefore,
>
> (iv) Platonism is true.

Once again, this argument is extremely compelling. We already argued for (i). But (ii) seems to follow immediately from (i). Think first of the sentence 'Mars is red'; this couldn't be true unless Mars existed. And likewise, given (i), '3 is prime' couldn't be true unless an abstract object existed, namely, the number 3. Finally, (iii) seems obvious. So our level-headed empirical semantic investigation seems to have led us to a crazy ontological thesis. We have two seemingly obvious premises--namely, semantic platonism and the truth of mathematics--and they lead to the crazy conclusion that there's a platonic realm of non-physical, non-mental, non-spatiotemporal objects. How did that happen?

Well, one analysis of how it happened is that premise (iii) is a lot more controversial than it seems. For given our platonistic semantics, the claim that mathematical sentences like '2 + 2 = 4' are literally true is tantamount to the claim that platonism is true. We can bring this point out very clearly by showing that one might also argue as follows:

> (i) Semantic platonism is true--i.e., ordinary mathematical sentences like '2 + 2 = 4' and '3 is prime' are straightforward claims about abstract objects (or at any rate, they purport to be about abstract objects). Therefore,
>
> (ii) Mathematical sentences like '2 + 2 = 4' and '3 is prime' could be true only if platonism were true--i.e., only if abstract objects existed. But
>
> (iii*) Platonism isn't true--i.e., there's no such thing as platonic heaven, and there are no such things as non-physical, non-mental, non-spatiotemporal abstract objects. Therefore,
>
> (iv*) Mathematical sentences like '2 + 2 = 4' and '3 is prime' are not true.

You might think this argument is just as compelling as the last argument. It too has extremely plausible premises and a crazy conclusion. But it's not clear which argument is better.

(We can call the view expressed in this argument fictionalism. But it's important to note that mathematical fictionalists do not think that mathematics is perfectly analogous to novel writing.

That's not the view. The view is simply that mathematical sentences aren't literally true because (a) they're supposed to be about abstract objects (i.e., semantic platonism is true), and (b) there are no such things as abstract objects. So a better name would be not-literally-true-ism. In any event, this view was first introduced by Field (1980), and it has been further developed by Rosen (2001), Yablo (2002), Leng (2010), and myself (1998).)

In any event, which of these two arguments should we endorse? Well, there's also a third argument here that's a bit safer than either of the first two arguments and that is, I think, very interesting. I would actually endorse this third argument. It goes like this:

> (i) Semantic platonism is true--i.e., ordinary mathematical sentences like '2 + 2 = 4' and '3 is prime' are straightforward claims about abstract objects (or at any rate, they purport to be about abstract objects). Therefore,
>
> (ii) Mathematical sentences like '2 + 2 = 4' and '3 is prime' could be true only if platonism were true--i.e., only if abstract objects existed. Therefore,
>
> (iii**) Either platonism or fictionalism is true.

Insofar as platonism and fictionalism are both crazy, we seem to have a purely empirical argument here for the claim that something crazy is going on in the philosophy of mathematics. If our empirical semantic theory is right, then our only options are platonism and fictionalism. And as far as I can see, there's no good reason for favoring either of these views over the other.

(I suspect that for a lot of mathematicians, the idea that '2 + 2 = 4' is false is pretty hard to swallow. If that's how you feel, then you can endorse platonism--though, for the life of me, I don't know how you could justify that claim. But perhaps it will give you some solace to learn that according to fictionalism--or at any rate, the best versions of fictionalism--it isn't just mathematics that turns out to be untrue. According to the version of fictionalism that I favor, empirical theories like Quantum Mechanics are untrue as well, because these theories refer to abstract mathematical objects. So mathematicians are no worse off in this regard than anyone else is. Now, maybe it bothers you to think that our mathematical and scientific theories are untrue. But it doesn't bother me. The trick is to notice that (a) according to fictionalism, our mathematical and scientific theories are virtually true, or for-all-practical-purposes true, or some such thing (because they're such that they would be true if there were abstract objects), and (b) if fictionalism is true, then it's this virtual truth, or for-all-practical-purposes truth, that's really important. Literal truth, on this view, just isn't very important; it isn't to be valued; and so it just doesn't matter if our mathematical and scientific theories aren't literally true.)

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