WHY IS IT PLAUSIBLE?

(Barry Mazur, JMM conference, Jan. 5, 2012)

$$(\mathbf{A}) \qquad \Longrightarrow \qquad (\mathbf{B})$$

(B) is plausible

————————————

We gain confidence in **(A)**

George Pólya (1887 -1985)

Mathematics and Plausible Reasoning

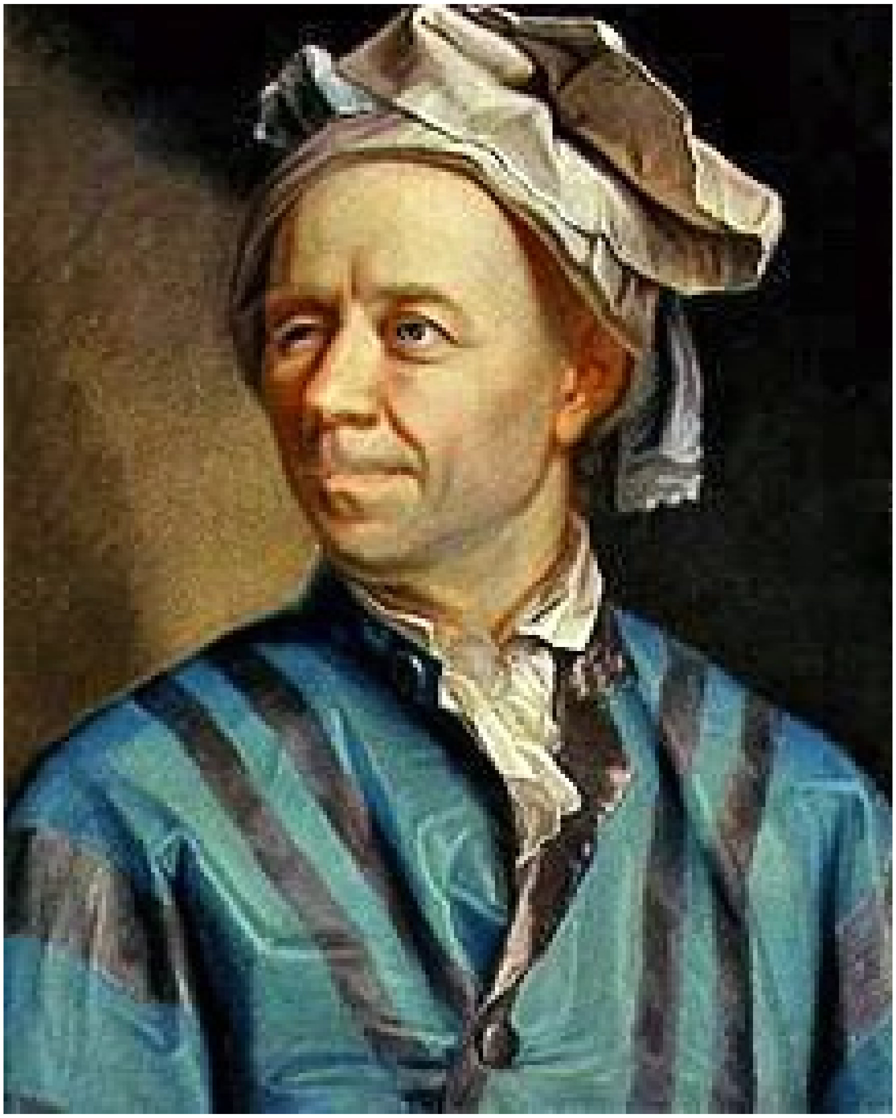
- Vol. I: Induction and Analogy in Mathematics
- Vol. II: Patterns of Plausible Inference

How do we gain confidence in mathematical guesses, before we actually prove them?

A personal list:

- *reasoning from consequence*,
- *reasoning from randomness*,
- *reasoning from analogy*.

Leonhard Euler (1707-1783)



I. Reasoning from Consequence:

If (**A**) *implies true things we gain confidence in* (**A**)

- Induction
- Experimental confirmation
- "Inferential fallacy"

Euler's Conjecture

Any number of the form $3 + 8n$ is expressible as a square plus twice a prime.

$$\boxed{\textbf{(A)} \qquad 3 + 8n \quad = \quad a^2 + 2p.}$$

Test it numerically:

$$11 = 1^2 + 2 \cdot 5$$

$$19 = 3^2 + 2 \cdot 5$$

$$27 = 1^2 + 2 \cdot 13$$

$$35 = 1^2 + 2 \cdot 17 = 3^2 + 2 \cdot 13 = 5^2 + 2 \cdot 5$$

$$\ldots$$

Why was Euler interested in this conjecture?

Assuming it, Euler could prove:

Any number is a sum of three trigonal numbers:

$$\textbf{(B)} \qquad n = \frac{x(x+1)}{2} + \frac{y(y+1)}{2} + \frac{z(z+1)}{2}$$

(**B**) is a special case of

Fermat's polygonal number "theorem,"

and

(**B**) was eventually proved by Gauss:

$$\boxed{\text{Eureka! num} = \Delta + \Delta + \Delta}$$

So. . . (??)

(inverted modus ponens)

$$(\mathbf{A}) \qquad \Longrightarrow \qquad (\mathbf{B})$$

(B) is plausible, thanks to Fermat

Euler gains confidence in **(A)**

??

We might think of the above diagram as one of the mainstays of the *calculus of plausibility*,

while modus ponens is key in the *calculus of logic*.

BUT, of course, there are vast differences

between these two brands of "calculus."

In the calculus of plausibility, our prior assessments are all important.

How *much* $(\mathbf{A})$ gains in plausibility, given that

$$\boxed{(\mathbf{A}) \Longrightarrow (\mathbf{B}) \text{ and } (\mathbf{B}) \text{ holds}}$$

depends on judgments about the relevance of $(\mathbf{B})$ vis à vis $(\mathbf{A})$.

It is often influenced by our sense of surprise that $(\mathbf{B})$ is true, if we are, in fact, surprised by it.

THE META-STABILITY OF PLAUSIBILITY

- **Reasoning by consequences** can decay under scrutiny!
- A tiny logical shift changes the calculus of plausibility:

(Hempel's Paradox)

All ravens are black

versus

No non-black object is a raven

Accumulating evidence for Riemann's Hypothesis:

- Find a (nontrivial) zero of the Riemann ζ-function and check that it actually lies on the line $Re(s) = \frac{1}{2}$, or:

- find a point s_0 in the complex plane that is (not a trivial zero and is) *off* the line $Re(s) = \frac{1}{2}$, and check that $\zeta(s_0) \neq 0$.

II. Reasoning from Randomness

We know all the relevant systematic constraints in the phenomena that we are currently studying, and . . . the rest is random.

Here's an Example! (A version of the ABC Conjecture)

Let a, b, c be a triple of positive integers. Consider the diophantine equation

$$A + B = C$$

where A, B, and C are positive integers and:

- *A is a perfect a-th power,*
- *B a perfect b-th power, and*
- *C a perfect c-th power.*

Let X be a large positive integer, and $N(X)$ be the number of solutions of our diophantine equation with $C \leq X$.

What can we say about the behavior of $N(X)$ as a function of the bound X?

To guess the answer we must:

(1) Deal with any “regularities” that we’re aware of; e.g. add the requirement that $GCD(A, B, C) = 1$.

(2) Assume that everything else behaves in an elementary random way.

There are:

$\sim\ X^{1/a}$ possible values of A less than X,

$\sim\ X^{1/b}$ possible values of B,

and $\sim\ X^{1/c}$ possible values of C.

So, working with numbers A, B, C less than X we see that we have

$$X^{\frac{1}{a}} \cdot X^{\frac{1}{b}} \cdot X^{\frac{1}{c}} = X^{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

shots at achieving a "hit," i.e., such that the value $A+B-C$ is zero.

But $A+B-C$ will range roughly (ignoring multiplicative constants) through X numbers, so the "chance" that we get a hit will be:

$$N(X) \sim \frac{1}{X} \cdot \text{the number of shots} \sim X^{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1}.$$

So, if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1$ is negative we arrive at the ludicrous expectation that $N(X)$ goes to zero as X goes to infinity, suggesting:

CONJECTURE: If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1$ there are only **finitely many solutions**.

Back to Euler's Conjecture

(and Reasoning from Randomness)

At least, (**A**) stands a chance:

$$\boxed{(\mathbf{A}) \qquad 3 + 8n \quad = \quad a^2 + 2p.}$$

$$X \quad \text{versus} \quad X^{\frac{1}{2}} \cdot \frac{X}{\log(X)}$$

III. Reasoning from Analogy

- Analogy by expansion
- Analogy as Rosetta stone

Analogy by expansion

More standard is to call it "generalization."

Enlarging a template.

It may have the appearance, after the fact, of being a perfectly natural "analytic continuation," so to speak, of a concept—such as the development of zero and negative numbers as an expansion of whole numbers, and from there: rational numbers, etc.

BUT it also may have, and retain, the shock value of a fundamental change. . .

Such as *Grothendieck topologies* that offer a radical refiguring of what it means to be a topology.

In contrast, we all are on the lookout for *incremental expansion* all the time.

Back to Euler's Conjecture and "analogy by incremental expansion:"

Consider

$$\boxed{(\mathbf{A}) \qquad 3 + 8n \quad = \quad a^2 + 2p.}$$

versus

$$\boxed{(\mathbf{A}') \qquad 3 + 8n \quad = \quad a^2 + \{(b+c)^2 + (b-c)^2\}.}$$

versus

$$\boxed{(\mathbf{B}) \qquad n = \frac{x(x+1)}{2} + \frac{y(y+1)}{2} + \frac{z(z+1)}{2}}$$

Analogy as Rosetta stone

(Much of current mathematics!)

André Weil's famous paragraph on analogy:

Nothing is more fruitful—all mathematicians know it—than those obscure analogies, those disturbing reflections of one theory on another; those furtive caresses, those inexplicable discords; nothing also gives more pleasure to the researcher. The day comes when this illusion dissolves: the presentiment turns into certainty; the yoked theories reveal their common source before disappearing. As the Gita teaches, one achieves knowledge and indifference at the same time.

IV. Variants of Plausible

Useful wedges. . .