Natural Logicism

by

Neil Tennant*

Department of Philosophy
The Ohio State University
Columbus, Ohio 43210
email tennant.9@osu.edu

October 2, 2011

Abstract

The aim here is to set out a philosophy, foundations and methodology for mathematics that can best be described as 'natural logicist'. It is what logicists might have arrived at, had they enjoyed the benefits of Gentzen's methods of natural deduction. Mathematical content, we argue, is best captured by rules set out in a natural deduction format, governing a primitive vocabulary of expressions belonging to a variety of syntactic types. The objects of each mathematical theory—for example, numbers in the case of arithmetic; points, lines and planes in the case of geometry—are to be treated as *sui generis*. The task is then to find suitable rules governing the basic functions and relations characteristic of the different mathematical domains.

*Earlier versions of this paper were presented at the Workshop on Reverse Mathematics in Chicago, in November 2009; and as an invited lecture to the Philosophy of Mathematics Special Interest Group of the Mathematical Association of America (POMSIGMAA) in Lexington, Kentucky, in August 2011. The author is grateful to both audiences for helpful questions. The current version is a working draft, and is neither polished nor complete. The author welcomes any comments from interested readers.

Contents

1 A brief historical background

There are two ways that logic serves mathematics.

1. Logic provides formal proofs that 'regiment' the informal proofs of ordinary mathematics.

2. (More controversial:) Logic furnishes definitions of the primitive concepts of mathematics, allowing one to derive the mathematician's 'first principles' as results within logic itself.

With (1), one is pursuing logic as proof theory. With (2), one passes from logic proper to *logicism*, which is a specific philosophical account of the nature of mathematical knowledge and its objects. We shall take these in reverse order.

1.1 Up to and including Frege

Logicism is the philosophical and foundational doctrine, concerning mathematics, that mathematical truth is a species of logical truth, and that mathematical objects are logical objects. The truths of mathematics are therefore *analytic* in Kant's sense—they are true *solely by virtue of the meanings of the linguistic expressions involved.* Another consequence of this view is that mathematical certainty is of a piece with certainty about logical truth. The same holds for necessity.

The doctrine of logicism had its first glimmerings in the writings of Dedekind, but it really only came to full flowering in the work of Frege. Their combined contributions represented a culmination of the trend, by their time well under way among leading mathematicians, towards the *arithmetization* of real (and complex) analysis. This trend had its beginnings in the even earlier works of Gauss and Bolzano. It came to maturity in the works of Cauchy and Weierstraß, and became the dominant paradigm in Western thought about the nature of mathematics. The leading idea of the arithmetizers was that the concepts and first principles of arithmetic and analysis are to be found in the human understanding, independently of its geometric intuitions concerning any spatial or temporal continua. Arithmetic and analysis are completely *conceptual* and *logical* in their axiomatic sources and in their deductive development.

Frege, as is well known, went in for overkill with the formal system that was to vindicate his logicism. He sought to unify all of arithmetic and analysis within a general theory of *classes*. Classes were supposed to be logical objects *par excellence*. The strategy was to define the natural numbers, say, as particular classes within a much more capacious universe of abstract, logical objects. Using the definitions, one would then derive the first principles of arithmetic (the Peano–Dedekind axioms, say) as theorems within the theory of classes. To that end one would exploit, ultimately, only the deeper underlying axioms governing classes themselves.

Among these axioms was Frege's ill-fated Axiom V—a principle of what is today known as 'naïve comprehension'. Corresponding to any property Φ, Frege maintained, there is the class of all and only those things that have the property Φ. In the language of second-order logic:

$$\forall\Phi\exists X\forall y(y \in X \leftrightarrow \Phi y).$$

Russell's famous paradox ensued. For Φy take $y \notin y$. One thereby obtains

$$\exists X\forall y(y \in X \leftrightarrow y \notin y).$$

Let r be such an X. So

$$\forall y(y \in r \leftrightarrow y \notin y).$$

But r is an object within the scope of this generalization. Instantiating with respect to r, one obtains

$$r \in r \leftrightarrow r \notin r.$$

But one can show in short order, within a very weak propositional logic, that any statement of the form

$$A \leftrightarrow \neg A$$

is inconsistent.[1] So Frege's Axiom V is inconsistent.

This simple formal discovery occasioned the 'crisis in foundations' early in the 20th century.

1.2 After Frege and up to Zermelo, but before Gentzen

Russell offered his own solution to the problem of his paradox, in the form of his *theory of types* (both simple and ramified). By stratifying the universe of objects into types, or levels, he sought to avoid the vicious circularity that he had diagnosed as the underlying problem with Fregean class abstraction. According to Russell, it should be illicit to define a class C in a way that

[1] We have been using the mathematician's 'relation-slash' notation $r \notin r$ instead of the logician's 'sentence-prefix' notation $\neg(r \in r)$.

involves generalizing about any range of individuals to which C itself would have to belong. Thus the notion of self-membership, along with non-self-membership, could not even be deployed.

This Russellian constraint on class abstraction, however, meant that for many a 'class abstract' of the form 'the class of all x such that $\Phi(x)$', the *existence* of such a class could not be guaranteed *as a matter of logic*. Instead, one would have to *postulate* that such classes existed. And this came to be regarded as detracting from their status as would-be logical objects, and revealing them instead as no more than mathematical posits. Their existence was once again a *synthetic a priori* matter, rather than one of analytic necessity and certainty.

Exactly why such classes would have qualified as logical objects courtesy of a single immensely powerful postulate (had it been consistent), but would *not* so qualify if their existence has to be secured in a more piecemeal postulational fashion, has never been clear to this writer. But that was the Achilles heel of Russellian logicism. The existential postulation present in his Multiplicative Axiom (nowadays known as the Axiom of Choice) and in his Axiom of Reducibility were seen as marks of the merely mathematical, albeit against the background of a much more capacious universe of abstract objects than just the natural numbers or the real numbers themselves.

For various methodological reasons, the theory of types fell out of favor as a foundational theory for mathematics, and was replaced by the newly emerging *set theory* due to Zermelo and Fraenkel. This displacement took place during the 1920s. The aim was still to unify all of mathematics, and to provide a capacious universe of abstract objects in order to do so. All the different mathematical theories would be interpretable within set theory, upon suitable identification of 'set-theoretic surrogates' for the objects studied by those theories. So, for example, the finite von Neumann ordinals can serve as set-theoretic surrogates for the natural numbers.[2] And $\wp(\omega)$, the power set of the set of natural numbers, is the set-theoretic surrogate for the real continuum.[3]

After the profound influence of the Séminaire N. Bourbaki in the early to mid-twentieth century and the 'new math' revolution that it occasioned, we are nowadays accustomed to conduct our higher mathematical thinking

[2]We do not say the finite von Neumann ordinals are *the* set-theoretic surrogates for the natural numbers, because of the well-known 'Benacerraf point' that there are other recursive progressions within the universe of (hereditarily finite) pure sets that could serve just as well—Zermelo's finite ordinals, for example.

[3]Cantor's theorem (that every set has strictly more subsets than members) has the special case that $\wp(\omega)$ has more members than ω does, hence is uncountable.

only in set-theoretical terms. But many of the 'textbook' results of advanced mathematics were formulated and proved (in accordance with the standards of informal rigor of the time) before set theory had reached maturity as the general 'carrier theory' for higher mathematics.

1.3 After Gentzen

It was only with the work of Gerhard Gentzen in the early 1930s (see [3]) that researchers in foundations were equipped with formal calculi of deduction that could do real justice to the actual structure of dependencies within mathematical proofs. What we have in mind here are the dependencies of conclusions upon both premises and assumptions that may have been made only 'for the sake of argument'. Examples of assumptions of the latter kind are *reductio* assumptions (assume φ; derive absurdity; conclude $\neg\varphi$, now independently of φ); assumptions for conditional proof (assume φ; derive ψ; conclude $\varphi \to \psi$, now independently of φ); and the case-assumptions for proof-by-cases (first assume φ in order to derive θ; secondly, assume ψ in order to derive θ; thence conclude θ from $\varphi \vee \psi$, independently of the case-assumption φ that appears in the first case-proof, and independently of the case-assumption ψ that appears in the second case-proof).

Looking back, it strikes one as quite extraordinary that the community of mathematical logicians took *so long* to discover the calculi of natural deduction (and the sequent calculi), once Frege, in 1879, had cracked the previously hidden grammatical code of multiply quantified sentences. It is extraordinary that Gödel, in 1929, could have demonstrated the completeness of first-order logic *before* Gentzen's natural formulation of it, when that logic was available only in the forms of the highly unnatural deductive calculi devised by Frege, by Hilbert, and by Russell and Whitehead.

The essential breakthrough of Gentzen's treatment was to treat each logical operator in isolation, with rules of its own, rules in which *only* that operator would explicitly feature. Moreover, the rules in question would deal only with a *single occurrence* (in dominant position) of the operator in question. The rule for reasoning *to* a conclusion with the operator dominant was called its *introduction* rule; while the rule for reasoning *from* a premise with the operator dominant was called its *elimination* rule.

The introduction and elimination rule for any logical operator have to be in a certain kind of *equilibrium*, an equilibrium that lends itself to an interpretation of the rules as forming the basis of an intelligible *social contract* between any responsible, rational and sincere speaker, and any responsible, rational and trusting listener.[4]

The equilibrium in question is explicated by the so-called *reduction procedures* for the logical operators. These procedures enable one to remove from a proof any sentence occurrence that stands both as the conclusion of an application of an introduction rule and as the major premise of an application of the corresponding elimination rule. Repeated application of the procedures will eventually turn the proof into one that is in *normal form*—essentially, one that is not eligible for any further application of the procedures.[5]

The reduction procedure for conjunction illustrates the foregoing ideas clearly. In the case of conjunctions $A \wedge B$

> (i) one who hears $A \wedge B$ sincerely asserted should be able logically to infer from it both A and B: for this is all the information that the asserter ought to have acquired before inferring *to* $A \wedge B$.
>
> (ii) one who undertakes to assert $A \wedge B$ should ensure that both A and B are indeed the case: for A and B are what any listener would be able logically to infer *from* $A \wedge B$.

This harmony between speaker's obligations, as in (ii), and listener's entitlements, as in (i), is brought out by the following two-part *reduction procedure* for $\wedge$:[6]

$$\dfrac{\dfrac{\begin{array}{cc}\Delta & \Gamma \\ \Pi & \Sigma \\ A & B\end{array}}{A \wedge B}}{A} \quad \mapsto \quad \begin{array}{c}\Delta \\ \Pi \\ A\end{array} \qquad\qquad \dfrac{\dfrac{\begin{array}{cc}\Delta & \Gamma \\ \Pi & \Sigma \\ A & B\end{array}}{A \wedge B}}{B} \quad \mapsto \quad \begin{array}{c}\Gamma \\ \Sigma \\ B\end{array}$$

Here, Π is a proof of the conclusion A from the set Δ of premises; and Σ is a proof of the conclusion B from the set Γ of premises. The sets Δ and Γ can in general be distinct; indeed, they can be disjoint.

---

[4] It has been Michael Dummett, especially, who has advanced this interpretation. The present author has also stressed the importance of such equilibrium for the evolution of logically structured natural languages. See [13] and [15].

[5] The normalization theorem is due to Prawitz. See [7].

[6] Here, for ease of illustration, we follow Gentzen and Prawitz in using $\wedge$-E in its *serial* form. Elimination rules can also be stated in a *parallelized* form, due to Peter Schroeder-Heister. See [8]. For the advantages of the parallelized rules in automated proof-search, see [16]. For their special role in analyzing *relevance* of premises to conclusions within proofs, see [18] and [24].

The unreduced proof-schema on the left in each case shows $A \wedge B$ standing *both* as the conclusion of ($\wedge$-I) *and* as the major premise of ($\wedge$-E). In other words, the operator $\wedge$ is introduced, and then immediately eliminated. The occurrence of $A \wedge B$ is *maximal*. The reducts to the right of each arrow respectively show that one cannot thereby obtain anything that one did not already possess.

Note also that each of the reducts on the right of the arrow $\mapsto$ has either Δ or Γ as its set of undischarged assumptions. Whichever one it is, it could well be a *proper subset* of the overall set $\Delta \cup \Gamma$ of undischarged assumptions of the unreduced proof-complex on the left. So with the reduction procedure for $\wedge$ we learn an important lesson: reducing a proof (i.e. getting rid of a maximal sentence occurrence within it) *can in general lead to a logically stronger result*. This is because when Θ is a *proper* subset of Ξ, the argument $\Theta : \varphi$ might be a logically stronger argument than the argument $\Xi : \varphi$. It *will* be a logically stronger argument if one of the sentences in $(\Xi \setminus \Theta)$—that is, the set of members of Ξ that are not members of Θ—does not itself follow logically from Θ. To summarize: by dropping premises of an argument, one can produce a logically stronger argument. And reduction can enable one to drop premises in one's proof of an argument. So reduction is a potentially *epistemically gainful* operation to perform on any eligible proof, for it can produce a logically stronger result.

Powerful, incisive and revolutionary though Gentzen's approach has since proved to be, it was, in its turn, curiously limited. It was restricted to just the universally acknowledged *logical* operators of first-order logic: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\exists$ and $\forall$. At exactly the same time (1934), Carnap published his *Logische Syntax der Sprache*, which offered an account of analyticity for languages in which all logico-*mathematical* operators could make similar contributions to the status of a sentence as analytically true (or analytically false). Carnap, however, did this by employing axiomatizations involving all the various logico-mathematical operators, co-functioning in grammatically complex axioms. His approach was therefore quite unlike that of Gentzen, which was *single-operator focused*. The 1930s let the tradition fall prey to an unfortunate methodological lacuna: a failure to generalize Gentzen's approach beyond the strictly logical operators of first-order logic. Proof theory was thereby deprived of a potentially fertile agenda: an investigation of the various forms that introduction and elimination rules might take, as it addresses rule-governed expressions whose rules are not quite so neatly classifiable as introduction and elimination rules. This is the case, for example, with families of 'coeval' and interdependent concepts of a nevertheless logico-mathematical kind. An excellent example of such a family is that of the *ordered pair* of any two things; the *first member* of any ordered pair; and the *second member* of the same. See [21] for further details. (Another novel feature of this example, and of other examples that could be given, is that the operators in question are *term-forming operators*. Gentzen had confined his study to *sentence-forming* operators.) Perhaps it was Tarski's theory of truth for formalized languages (see [11]) that deflected interest away from further development of this essentially *inferentialist* approach to the meanings of logical and mathematical operators.

2 Foundations of Mathematics, in its mature phase

The area of study known as 'foundations of mathematics' aims to classify various mathematical theories according to their various systematic properties, and to study important relations amongst them. It also seeks a single *unifying*, over-arching theory that can accommodate all those more 'provincial' theories, upon suitable interpretation of the latter in the former.

2.1 The foundationalist's main aims

Given the usual 'branches' of mathematics, such as number theory, geometry (Euclidean and non-Euclidean), topology, etc., the foundationalist typically seeks to

1. identify their primitive concepts and first principles;
2. formulate a logic that applies to all of them;
3. investigate relationships among various theories, such as
   (a) T conservatively extends T';
   (b) If T is consistent, then T' is consistent;
   (c) T is interpretable in T';
4. provide a unifying theory with as few primitives as possible, in which all of the theories can be interpreted;
5. investigate properties of theories such as
   (a) T has a (countable) model;
   (b) T is κ-categorical, i.e. T has exactly one model, up to isomorphism, of cardinality κ;

(c) T is complete; and
(d) T is decidable.

2.2 The languages peculiar to different branches of mathematics

Different branches of mathematics, as any practising mathematician knows, are distinguished by their different stocks of concepts—which means that they are distinguished, in turn, by their 'extra-logical' *vocabularies* of expressions. These expressions can be simple names, function signs and/or predicates; and they can also be more complicated expressions, such as variable-binding operators. We offer here just four illustrations of the respective stocks of such primitives for different branches of mathematics.

- The language of arithmetic contains the primitive expressions 0, s, $+$, $\times$, $^\wedge$ (exponentiation)
- The language of 3-D projective geometry contains the following primitive expressions:
  - the *sortal predicates* '. . . is a point', '. . . is a line', '. . . is a plane';
  - the binary relation '. . . is included in (or lies on) —'; plus
  - the *incidence operators*
    * the point of intersection of line λ and plane Π
    * the point of intersection of the two co-planar lines λ and λ'
    * the line of intersection of the two planes Π and Π'
    * the line determined by the two points π and π'
    * the plane determined by the two intersecting lines λ and λ'
    * the plane determined by the line λ and the point π not on λ
- In addition to the primitives of arithmetic, the language of real analysis calls for
  - a notation for functions: $\lambda x.f(x)$; and
  - the ordering relation $<$.
- The language of set theory contains but one binary predicate, $\in$;[7] and, if one wishes, the set-abstraction operator $\{x| \ldots x \ldots\}$. (The latter is useful but not essential for the development of set theory.)

[7] We concede that it is quite extraordinary that all of mathematics can be obtained using only a single binary predicate.

3 The foundational project of Natural Logicism

It is interesting to inquire about the basic concepts and pre-set-theoretic, 'native' intuitions of the mathematicians who were able to formulate and prove important results that have only subsequently acquired their predominantly set-theoretical trappings. We have in mind here not the kind of not-altogether-trustworthy geometric intuitions against which real analysis[8] (so it was thought) had to be guarded via the arithmetization undertaken by Cauchy and Weierstraß. Rather, we have in mind the *analytic*[9] intuitions[10] of the competent mathematician, which, when clear and distinct, betoken a thorough grasp of the mathematical concept(s) involved. A case in point would be the intuition that the natural numbers obey the Principle of Mathematical Induction. (*Pace* Poincaré, we consider this intuition to be *analytic*, not synthetic. Its analyticity can be exhibited by furnishing it with a constructive logicist derivation using only rules that are analytic of the notions 'number of Fs', 'successor' and 'zero'.[11])

The aforementioned general inquiry is one on which the present author has been engaged for some time. It has led to a 'natural-logicist' reformulation of arithmetic,[12] projective geometry,[13] Euclidean geometry, real differential calculus,[14] the theory of higher infinities,[15] and set theory itself[16] (as just one among other branches of higher mathematics, rather than as an all-embracing foundational theory).

In the course of laying a natural-logicist foundation for each mathematical discipline, it has proved imperative also to achieve clarity about the norms of logical inference within a free logic with abstraction operators.

[8]Here, 'analysis' is meant in the mathematical sense—the study of real numbers and functions of reals.

[9]Here, 'analytic' is meant not in the mathematical sense mentioned in footnote 8, but in the Kantian sense, as arising from the meanings of the words involved.

[10]Here, 'intuitions' is used in the sense of ordinary mathematical parlance, and not in the special Kantian sense of 'telling us something informative about the world', which Kantians regard as the contradictory of the Kantian sense of 'analytic'!

[11]See [14], ch. 25, 'On deriving the basic laws of arithmetic: or, how to Frege-Wright a Dedekind-Peano'.

[12]*Ibid.*

[13]See [20].

[14]See [22].

[15]See [23].

[16]Just as the Dedekind-Peano axioms for arithmetic are *derived* as non-trivial results within a deeper logicist account of number, so too the Axiom of Extensionality of set theory is derived non-trivially from the deeper principles laid down in the logicist account of the set-abstraction operator. See [19] for details.

The present author has dealt successively with the description operator, the number-abstraction operator and the set-abstraction operator.[17] With the appropriate logic formulated, one can turn one's attention to the concepts and operations specific to each mathematical discipline in turn.

- A logical investigation of orderly pairing affords a logicist treatment of addition and multiplication in arithmetic.[18]
- A natural-deduction investigation of the incidence operators in projective geometry affords a logicist treatment of the foundations of that discipline within which its famous *Principle of Duality* is beautifully clarified:

> Given any proof Π of a theorem φ, the dual result φ' is proved by the dual proof Π', obtained by merely interchanging 'point' and 'plane' (in the 3-D case), likewise interchanging pairs of the incidence operators mentioned above, and switching the arguments of all occurrences of the incidence relation.[19] One otherwise *leaves unchanged* the macro-structure of Π (i.e. its pattern of applications of rules of inference) in thus passing to its dual Π'.[20]

- A natural-deduction investigation of functions *sui generis* (and *not* by way of set-theoretical surrogacy) leads to an elegant treatment of derivatives of real functions, which hews to the basic intuitions of the student of calculus.[21]

3.1 The aims of Natural Logicism

3.1.1 Rigorous regimentation of mathematical reasoning

Natural Logicism aims rigorously to regiment reasoning within any given branch of mathematics by using a system of appropriate natural-deduction rules. This is what justifies calling this version of logicism 'natural'.

---

[17] See [12], ch. 7, and [19].

[18] See [21].

[19] In the 2-D case, one interchanges 'point' and 'line' within the proof Π, interchanges appropriately different pairs in the dimensionally reduced set of incidence operators, and switches the arguments of all occurrences of the incidence relation.

[20] See [20].

[21] See [22].

3.1.2 Identification of conceptual primitives, *and defined concepts*, with an eye to actual mathematical practice

The defined concepts should be manageable, fruitful, and of wide application. They should help to *atomicize* the reasoning.

3.1.3 Formulation of introduction and elimination rules for conceptual primitives and for concepts defined in terms of them

Such rules pin down the concepts in question. This is what justifies use of the label 'logicism'.

Judicious choice of definitions, in which the definienda are furnished with introduction and elimination rules, enable one to minimize the logical complexity of sentences appearing in the formal proofs provided as regimentations of passages of informal mathematical reasoning.

3.1.4 Formal proofs should be homologues of informal ones; formalization should merely 'supply missing details'.

The early forms of logicism tended to obscure the virtues of logical rigor (in the regimentation of mathematical proofs) because they were tied to a quite orthogonal project. This was the project of trying to furnish an all-embracing, over-arching theory of classes or theory of types. The universe of discourse of the sought unifying theory, it was hoped, would accommodate (through appropriate surrogates) all the various kinds of mathematical objects that different mathematical theories are 'about'.

This Fregean and Russellian bent had the consequence that Logicism, as a philosophy and foundations for mathematics, appeared to be over-ambitious. Yet Logicism can and should be prosecuted without any concern for the unification of mathematics via class theory or set theory or category theory or the theory of types (to name the most important 'unifying theories' on offer). A logicism worthy of the name could confine itself to simply making existing proofs in the main corpora of rigorous[22] but informal mathematics, *perfectly* rigorous because completely formal and symbolic.

[22]Here we mean 'rigorous' to be understood in the usual way that a well-trained mathematician understands it. All main steps are explicitly indicated. Appeals to intuition are made only when the writer and the reader can be expected to know how to eliminate them in favor of more rigorous symbolic reasoning.

3.1.5 Formal proofs should be in normal form (when all their undischarged assumptions are declared).

This methodological constraint forces one to make judicious choices of lemmas interpolated between one's mathematical axioms and the theorems that one seeks to derive from them. It obliges one also to deploy a logic that enables one to make the kind of *deductive progress* that is evident throughout mathematics. The standard systems of classical logic and intuitionistic logic provide a rule of 'unrestricted Cut', which is designed to ensure that proofs can be 'accumulated': if one has proved lemma φ from axioms Δ, and has also proved theorem ψ from lemma φ plus further axioms Γ, then one has *ipso facto* proved theorem ψ from the combination of axioms $\Delta \cup \Gamma$:

$$\frac{\Delta : \varphi \qquad \varphi, \Gamma : \psi}{\Delta, \Gamma : \psi}$$

It is not necessary, however, to have a rule of Cut in such an unrestricted form in order to ensure deductive progress in mathematics. Indeed, it is not necessary to have a rule of Cut at all, as part of one's logical system. All that is needed, rather, is the truth of the following:

> There is an effective binary operation [,] on proofs such that given any proof Π of $\Delta : \varphi$, and given any proof Σ of the sequent $\varphi, \Gamma : \psi$, the proof $[\Pi, \Sigma]$ proves either ψ or $\perp$ from (some subset of) $\Delta \cup \Gamma$ (that is, $[\Pi, \Sigma]$ proves either a sequent $\Theta : \varphi$ or a sequent $\Theta : \perp$, for some subset Θ of $\Delta \cup \Gamma$).[23]

A remarkable feature of the introduction and elimination rules for the standard logical operators is that (with the elimination rules stated in their 'parallelized' form) one can insist that only normal-form proofs count as proofs, and still secure the truth of the displayed claim. Normality, moreover, is guaranteed by a beautifully simple expedient: major premises of eliminations must 'stand proud', with no proof-work above them. A major goal for the research program of Natural Logicism is to preserve these metalogical features of first-order logic when we extend it by adopting our envisaged introduction and elimination rules for more peculiarly *mathematical* notions.

[23]For fuller discussion of how this 'restricted transitivity' principle fully serves all the mathematician's needs, see [24].

3.1.6 Formalization should reveal points of non-constructivity, impredicativity, 'purity' etc.

This is one of the less appreciated benefits of full formalization of mathematical proofs. It enables the maturing mathematician to become aware of which steps of reasoning might be especially controversial or methodologically significant.

3.1.7 Treat the objects of the theory as *sui generis*, rather than as surrogate objects within a 'more foundational' theory such as set theory.

As remarked by Harrington, Morley, Ščedrov and Simpson in [5] at p. vii:

> ... ZFC ... is not appropriate ... for a more delicate study of the nature of mathematical proof. Standard mathematics is not inherently or peculiarly set-theoretic.

This remark was intended to set the stage, however, for their subsequent explanation of how arresting it was that Friedman had been able to demonstrate *necessary* uses of abstract set theory in order to prove results in 'relatively concrete mathematical situations' (*ibid.*, p. viii). What that means, however, is that the concrete result in question (φ, say) is provable in ZFC plus some large cardinal axiom, and in turn implies (*modulo* some weak base theory, such as EFA)[24] the consistency of ZFC plus all smaller large-cardinal axioms. *If one's main concern is to calibrate the consistency strength* of a particular concrete-looking conjecture φ in this way, then of course it behooves one to translate both φ and the 'native' axioms of the theory T (to which φ might or might not belong) into the language of set theory, so that the calibration can proceed. *If, however, one's main concern is to clarify the logical structure of the reasoning by which all the known results of the 'native' theory have been established*, then it is better to *eschew* the set-theoretical trappings that help only with the calibration question, and deal with T directly, natively, *sui generis*.

Mathematical theories are learned, developed and communicated 'natively'. Each theory has its own special stock of concepts; and is 'about' its own special kinds of mathematical object. The early proofs by great expositors of these theories treat these objects as *sui generis*, without presenting them as complicated sets drawn from the cumulative hierarchy of pure sets. The 'Bourbakization' of mathematics—the re-definition of all the

[24] EFA is exponential function arithmetic.

concepts of different branches of mathematics in terms of sets alone—makes it harder for a beginner to understand what any particular mathematical theory is *about*. It makes mathematics, which is already abstract enough, seem *utterly* abstract, to the point of enjoying no enlivening or illuminating connection whatsoever with any other area of human thought—be it physics, computer science or economics.

3.1.8 Explain how a given branch of mathematics is *applicable* (if it is).

The more searingly abstract a presentation one provides for a mathematical theory, the more difficult it becomes to explain how it is in the very nature of the mathematical objects concerned that one's theory about them can be applied in reasoning about real-world phenomena and the regularities that underly them. Ironically, it was Frege who made the most of the requirement that such pplicability be explained—and who then did the most damage to that very prospect.[25]

3.2 Some consequences of these aims

1. One attends more carefully to what is really 'built in' to a (defined) concept, as opposed to what is assumed in the hypotheses for one's reasoning.

2. One 'carves informal proofs at their joints'. *Regimentation is anatomization!*

3. One can more easily motivate the study of formal proofs for practising mathematicians.

4. One can devise proof-search strategies in automated or interactive theorem-proving that are tailored to the branch of mathematics in question.

5. One can address the issue of analytic v. synthetic truth in mathematics with sharper tools at one's disposal.

6. Occasionally one detects a deeply hidden fallacy in even the best extant texts.

[25] This is argued at greater length in [26].

3.3 Some prime–facie problems for the pursuit of these aims

1. One has to introduce an indefinite number of 'pasigraphically primitive' expressions (for all the defined terms).
   Answer: So what? We are thereby just mirroring the continuing development of informal mathematics, with its ever-increasing stock of defined concepts. The alleged problem is mitigated, indeed eliminated, by the fact that these defined concepts are always *finite in number.*

2. How can one tell whether the formal theory is consistent, or what its consistency strength is?
   Answer: Via mutual interpretation with alternative formalizations (which can of course be set-theoretic) whose consistency-strength has already been calibrated.[26]

3. The sheer grunt-work involved in thus 'perfecting' mathematical texts might be off-putting to [all/nearly all/most/many/several/a few... ?] mathematicians and/or foundationalists.
   Answer: *Der Teufel liegt im Detail!* Welcome to the task of fully Fregean foundations. At least it affords a prospect of gainful employment for logicians undertaking to do what logicians *ought* to do—which is to clarify the structure of mathematical reasoning at the most refined possible level of symbolic detail.

The question naturally arises, for any mathematical theory T: how far might this natural-logicist approach be extended to T? Could T be laid out in its own 'native' terms, shorn of the specifically set-theoretic notions that are employed in contemporary treatments in textbooks? Could one avoid the 'ontological riches' of a set-theoretic foundation, by helping oneself only to what is specifically needed, both conceptually and ontologically, in order to attain the results one is after?

[26]One of Friedman's remarkable results is that mutually interpretable theories have the same consistency strength *and conversely.* For exposition, see [10]. More recently, Friedman has written (see [2])

> The striking observation is that one finds a remarkable linearity [of consistency strengths]. This linearity is found not only with finitely axiomatized systems ... but with the non finitely axiomatized systems such as PA and ZFC. This is perhaps the most intriguing, thought provoking, fundamental, and deep phenomenon in the whole of the foundations of mathematics.

See §6 below for more details about this linear hierarchy of consistency strengths.

4 The development of Natural Logicism so far

In [12], ch. 7, a natural-deduction system for free logic was formulated, which could accommodate the potentially non-denoting terms that occur in mathematical theorizing. In [14] a general account was given of so-called 'transitional atomic logic'. A neo-logicist project was then pursued: a constructive and relevant foundation for arithmetic was set out, taking the number-term forming operator $\#x(\ldots x\ldots)$ as primitive. Meaning-determining introduction and elimination rules for # and the successor function were formulated, and the Dedekind-Peano axioms for arithmetic were derived for the language of 0, s and #.

The same project was extended, in [21], in order to deal with addition and multiplication. The concern was to elucidate the application of natural numbers to count finite collections, and to reveal a way in which essentially first-order resources could suffice for this purpose. The extra first-order resources affording a neo-logicist treatment of addition and multiplication were the rules for the logic of orderly pairing.

In [17] it was argued that the formulation of mathematical theories in terms of introduction and elimination rules for the main logico-mathematical operators furnished a principled basis for drawing an analytic/synthetic distinction within those mathematical theories. The operators in question are *term-forming* operators, not sentence-forming ones. Hence the natural-deduction paradigm of introduction and elimination rules (for connectives and quantifiers) had to be extended in order to deal with them. The rules for any term-forming operator have to characterize conditions for its introduction and elimination when it is dominant on one side of an identity statement.

For example, the 'logic' of sets that is generated by such introduction and elimination rules for the set-abstraction operator $\{x \mid \ldots x\ldots\}$ yields a body of analytic results, including the so-called axiom of extensionality—a body of theorizing that Quine once called 'virtual set theory'. On the synthetic side would be existential claims such as 'ω exists'.[27] In [19], a general account was provided of the logic of abstraction operators, including set-

[27] For the reader not versed in modern set theory: ω is defined to be the set of all finite von Neumann ordinals, which are the usual set-theoretic surrogates for the natural numbers. Corresponding to the natural number 0 is the empty set $\emptyset$. Corresponding to the natural number 1 is the singleton of 0, namely $\{\emptyset\}$. In general, corresponding to the natural number $n+1$ is the set $\{0,\ldots,n\}$. In this way, the 'less than' relation $<$ among natural numbers corresponds conveniently to the relation $\in$ of set-membership. Moreover, n is the cardinal number of a set X by virtue of there being a 1-1 mapping from X onto n.

abstraction. In [25] the method of formulating introduction and elimination rules was applied to the fusion operator in mereology, with systematic and simplifying effects.

Given that the general method of introduction and elimination rules could be claimed to have dealt, with some measure of success, with the notions of set, number, and part-whole, the neo-Kantian question arises: could this method bear fruit when applied to geometry?

In search for an answer to this question, [20] focused on the simplest and most elegant kind of geometry—synthetic projective geometry in both two and three dimensions (where 'synthetic' is used here in the geometer's sense). The logico-philosophical project just described is thereby extended beyond the author's earlier concerns with set, number, and the relations of parts to wholes.[28] [20] examined how much of the 'geometry of incidence' can be captured with simple and elegant rules for the introduction and elimination of the 'incidence operators'; and how much of projective geometrical theorizing depends on existential postulation that would have to be described as synthetic (where 'synthetic' is now used in the meaning-theorist's sense). The aim was to dig more deeply than even the most rigorously-minded geometers are wont to do, in order to furnish a logical foundation that will yield conventional axiomatizations as by-products. We proceeded as best we could at first order, avoiding recourse to any higher-order or set-theoretic or mereological conceptions of geometrical entities.

Various branches of pure mathematics, such as arithmetic, different geometries, set theory etc. have been axiomatized by the pure mathematicians who practice in those fields. These mathematicians are interested first and foremost in the abstract *structures* formed by the mathematical objects under investigation, even when the intention is to try to characterize the structure in question up to isomorphism. Questions of applicability are usually set to one side, as are questions concerning the ultimate logical foundations of that branch of mathematics within rational thought as a whole. One of the (perhaps unintended) consequences of this 'pure isolationist' approach is that axioms are chosen with a pragmatic eye on how quickly they can yield desired consequences, and how readily they will be accepted (without proof) by the intended audience. Both consistency and certainty are desiderata, to be sure; but pragmatic compromises are also struck in pursuit of both brevity of proof and power of single axioms.

This means, in the case of some of the traditional axiomatizations of

[28] Reference was made, however, in [17] at pp. 301 and 446, to an earlier stage of the work here decribed on natural foundations for projective geometry.

different branches of geometry, that there is a trade-off between the length of axioms and their number—usually increasing the former and decreasing the latter. The axioms eventually chosen serve mainly as convenient starting points for deductions, provided only that they will be accepted as true of the intended subject matter. There is no uncompromising concern, on the part of practising mathematicians, to ensure that all the axioms laid down are conceptually basic, or—even better—*analytic* of the concepts involved. Nor is there any concern to keep the axioms within some tightly constrained syntactic class, involving, say, a minimal number of quantifier alternations.

[20] departed rather self-consciously from this established precedent in mathematical axiomatization. Rather than stating *axioms*—which are (usually complex) sentences of a formal language—we stated *transitional atomic rules of inference*. These are rules of inference, in natural-deduction format, in which only atomic sentences feature. Some of them may contain parameters, thereby enabling one to express existential import—but still the only sentences in view are atomic. Secondly, we stated a great many rules, arranged, as far as possible, in thematically coherent groups. Our basic methodological principle was, and is: state more simply and more frequently, rather than less simply and less frequently. Fundamental principles of geometry should be like so many little ants, making for a supple organic whole, rather than like heavy foundation stones that are difficult to put in place.

5 What is distinctive about projective geometry?

The general philosophical reader may not be *au fait* with the ways in which projective geometry differs from affine geometry or from Euclidean geometry. So a few words of explanation will be in order. The following method of comparing and contrasting geometries is that of Klein's well-known *Erlanger Program*.

Each geometry is based on notions (primitive or defined) that *remain invariant under a distinctive group of transformations* of the space in question. The transformations form a group because the result of performing any two licit transformations in succession is a transformation that is itself licit. In the case of Euclidean geometry, the transformations are *distance-preserving*; they can be generated by a combination of (rigid) translations, rotations and/or reflections. Under all such transformations, the following features of geometrical figures will be invariant: incidences and tangencies; distances, ratios of distances, and equidistances; areas and volumes;

betweennesses; angles and perpendiculars; circles; parallels. So Euclidean theorems are allowed to employ any of these notions.

In affine geometry, by contrast, the transformations in question are *linear*: those stretchings and/or shrinkings in different directions that can deform squares into parallelograms, circles into ellipses, cubes into parallelopipeds, and spheres into ellipsoids. Under linear transformations, congruent Euclidean figures will remain at best *similar* figures. No longer will distances, ratios of distances, equidistances, areas, volumes, angles, perpendiculars or circles be invariant. So no affine theorem can employ these notions. But affine theorems can still employ the notions of incidences and tangencies, betweenness, and parallels. For these are invariant under linear transformations of space.

In projective geometry, the transformations in question are projections. These are best illustrated in the two-dimensional case. Imagine two planes, not necessarily parallel to one another, and a *point of central projection* not on either of them. Imagine a geometrical figure in the *original plane* (say, a triangle *abc*) being projected by straight lines through the point of projection onto its image in the *copy plane*. Note that the line from the point of central projection might pass through a source-point in the original plane before it hits the corresponding image-point in the copy plane; but that there can also be cases where the line from the point of central projection passes through the image-point in the copy plane on its way to the source-point in the original plane.

What geometrical features will remain invariant under such projections? The answer is: but a meager stock. Straightness of lines will survive. So too will incidences and tangencies; hence also collinearities of more than two points, and concurrencies of more than two lines. But when a point x is *between* points y and z on some line in the original plane, that need not be the case with their respective images x', y' and z' in the copy plane. And when a line k is parallel to line l in the original plane, that need not be the case with their respective images k' and l' in the copy plane.

The projective geometer has to exercise ingenuity to discover more abstract features of various geometric figures—features such as harmonic cross-ratios, or separation of one pair of points by another pair—that will be projectively invariant, and in terms of which interesting theorems may be stated and proved. Projective transformation is so far-reaching that a circle can be transformed into an ellipse, or a parabola, or an hyperbola, depending on its relation to the point of projection and the copy plane. In projective geometry, accordingly, there is but one notion of conic, uniting the three kinds of conic section familiar from the Euclidean case.

There is something seductively *ascetic* about projective geometric truths. They are formulated in terms of incidences alone, and are devoid of any concern for distance, angle, parallelism or betweenness. Projective geometry might be thought of as the bare logic of criss-crossing.

From a logical point of view, there is something seductively *aesthetic* about projective geometric truths. First, projective space is *incidence-replete*. Any two co-planar lines meet in a point—even if it has to be a 'point at infinity'. Likewise, any two planes meet in a line—even if it has to be a 'line at infinity'. This means that the logical rules for incidences can be framed in a smoothly exceptionless way. Secondly, every projective geometric truth has a *dual*—in two dimensions or in three dimensions, as appropriate. In two dimensions, for example, any two points determine a unique line containing them. Dually: any two lines determine a unique point that they contain. In three dimensions one has the same phenomenon, albeit with different 'switchings' or inter-substitutions of key notions. For example, any plane and any line not in it determine a unique point in that plane and on that line. Dually: any point and any line not containing it determine a unique plane containing that point and that line. (In three dimensions, one dualizes by interchanging 'point' and 'plane', while leaving 'line' undisturbed.)

The natural-deduction treatment proposed in [20] for projective geometry sought to capitalize on, and to highlight, both the existence of extremal elements (points and lines at infinity) and the inherent dualities in two dimensions and three dimensions respectively. It was a modest initial foray, by a proof-theoretically minded logician, into the theory (or logic?) of synthetic projective space.

It remains to be seen whether and to what extent the techniques developed in [20] for projective geometry might be adapted so as to yield similar systematic treatments of affine and/or Euclidean geometry. The author envisages an extension of the natural-deduction rules framed in [20] so as to avoid reliance on extremal elements and to cope with the newly admissible notions such as parallelism (in the affine case) and equidistance and betweenness (in the Euclidean case). The challenge will be to show how *predicates* (not: term-forming operators) can admit of similar logicist-inferentialist treatment.

6 The Linear Hierarchy of Consistency Strengths

We furnish here an account of the linear ordering, by consistency strength, of various mathematical theories of interest to the foundationalist. Consistency strength increases as one goes down the rows in the tables below. Theories with the same consistency strength are listed on the same row, but perhaps in different columns. The columns house the different kinds of the theories in question (1st-order arithmetics; 2nd-order arithmetics; higher-order arithmetics; 1st-order set theories; 2nd-order set theories). $T \approx T'$ means that the theories T and T' have the same consistency strength.

The calibration of consistency strengths is made possible by 'uniformizing' the ontology of the mathematical theories being compared. The natural numbers are taken as given, as forming one sort; and then the question becomes which *sets* of natural numbers one should acknowledge, as forming a second sort. The stronger the commitment on this second score, the greater the consistency strength of one's theory. Brilliant though the work has been, in comparing various theories when cast in these terms, one cannot help remarking how 'unnatural' that re-formulation is, when one considers the methods of coding employed in order to define negative integers, rational numbers, real numbers, etc. out of the naturals.

Word of warning: The terminology 'second-order' is well established among those foundationalists who study these matters; but the reader must not casually assume that the label is earned (or incurred) by quantification that is genuinely second order, that is, quantification over properties of, and/or relations among, ground-level individuals. Rather, the 'second-order' entities are ground-level entities of a second sort, so that one is dealing with a two-sorted, *first-order* system. Entities of the first sort can bear the membership relation to entities of the second sort; but entities of both sorts are nevertheless ground-level individuals—as indeed sets themselves are, within the context of first-order set theory.

Sources: We draw here on Table E: Twenty Milestones on the Fundamental Series, at pp. 220–1 of [1]; as well as on [2]. The interested reader will also find a wealth of information in [4], [6], and [9].

Note: P^2 is also called Z^2 (a.k.a. *analysis*) by those working on so-called second-order subsystems of arithmetic. We use Burgess's choice of P here (after <u>P</u>eano) in order to avoid confusion with <u>Z</u>ermelo set theory.

Colors: We have highlighted the best-known arithmetics, the most important systems of reverse mathematics, and the most important set theories.

Table 1: Linear Hierarchy of Consistency Strengths

1st-order arithmetics	2nd-order arithmetics	1st-order set theories
Q (Robinson arithmetic) $\approx$ Q_2(=$I\Delta_0$; $I\Sigma_0$; $I\Pi_0$; Polynomial Function Arithmetic; Nelson–Wilkie Arithmetic) Q_3(=$I\Delta_0$(exp); Kalmar Arithmetic; Exponential Function Arithmetic) Q_4(=$I\Delta_0$(sup); Superexponential Function Arithmetic; Gentzen Arithmetic) $\vdots$ Primitive Recursive Arithmetic $\vdots$		
Q_ω (Grzegorczyk Arithmetic) $\approx$ $I\Sigma_1$ (Parsons Arithmetic) $I\Sigma_2$ (Ackermann Arithmetic) $\vdots$	RCA_0 $\approx$ WKL_0	
PA (= P^1; Q_2 + Ind. for all formulae; Peano Arithmetic);	ACA_0	$Z^- - \infty$

Table 2: Linear Hierarchy of Consistency Strengths, cont.

1st-order arithmetics	2nd-order arithmetics	1st-order set theories
	$\mathrm{ACA}_0 + \forall n \forall x \exists!\, \mathrm{TJ}(n, x)$ ACA $\mathrm{RCA}_0 + \exists!\, \mathrm{TJ}(\omega)$ $\mathrm{ACA}_0 + \exists!\, \mathrm{TJ}(\omega)$ $\mathrm{ACA} + \exists!\, \mathrm{TJ}(\omega)$ $\mathrm{ACA}_0 + \forall x \exists!\, \mathrm{TJ}(\omega, x)$ $\mathrm{ACA}_0 + \{\forall x \exists!\, \mathrm{TJ}(\alpha, x) \mid \alpha < \omega^\omega\}$ $\mathrm{ACA}_0 + \forall \alpha < \omega^\omega\, \forall x \exists!\, \mathrm{TJ}(\alpha, x)$ $\mathrm{ACA}_0 + \forall x \exists!\, \mathrm{TJ}(\omega, x)$ $\mathrm{RCA}_0 + \exists!\, \mathrm{TJ}(\omega^\omega)$ $\mathrm{ACA}_0 + \exists!\, \mathrm{TJ}(\omega^\omega)$ $\mathrm{ACA}_0 + \forall x \exists!\, \mathrm{TJ}(\omega^\omega, x)$ $\mathrm{ACA}_0 + \{\forall x \exists!\, \mathrm{TJ}(\alpha, x) \mid \alpha < \epsilon_0\}$ Δ^1_1-CA $\mathrm{RCA}_0 + \exists!\, \mathrm{TJ}(\epsilon_0)$ $\mathrm{ACA}_0 + \exists!\, \mathrm{TJ}(\epsilon_0)$ $\mathrm{ACA} + \exists!\, \mathrm{TJ}(\epsilon_0)$ $\mathrm{ACA}_0 + \forall x \exists!\, \mathrm{TJ}(\epsilon_0, x)$ $\{\mathrm{ATI}(\alpha) \mid \alpha < \Gamma_0\}$ ATR_0	

Table 3: Linear Hierarchy of Consistency Strengths, cont.

1st-order arithmetics	2nd-order arithmetics	1st-order set theories
	ATI($< \Gamma_0$)	
	ATR	
	Π^1_2-TI_0	
	Π^1_2-TI	
	TI	
	ID_2	
	$ID_{<\omega}$	
	Π^1_1-CA_0 ($\approx \Pi^1_1$-Frege Arithmetic)	
	Π^1_1-CA	
	Π^1_1-CA + TI	
	Π^1_1-TR_0	
	Π^1_1-TR	
	Π^1_2-CA_0	Z^-_1
	Π^1_2-CA	
	Π^1_2-CA + TI	
	Π^1_3-CA_0	Z^-_2
	$P^2 = PA^2$ (a.k.a. Z_2, $\approx$ Frege Arithmetic)	$Z^- \approx ZF^-$

Table 4: Linear Hierarchy of Consistency Strengths, cont.

Higher-order arithmetics	1st-order set theories	2nd-order set theories
$\mathrm{P}^3 = \mathrm{PA}^3$	$\mathrm{ZF}^- + \wp(\omega)$	
$\mathrm{P}^4 = \mathrm{PA}^4$	$\mathrm{ZF}^- + \wp^2(\omega)$	
P^ω = PM (type theory)		
	Weak Zermelo	
	Z (Zermelo)	
	ZC ≈ Z + V=L	
	ZC + $\forall\alpha < \omega_1\, \exists!\, V_\alpha$	
	KP($\wp$)	
	ZFC ≈ ZF + V=L	NBG
		ZFC^2 = MK
	ZFC + Inaccessible	
	ZFC + Strongly Inaccessible	
	ZFC + Strongly Mahlo	
	ZFC + $\{\exists!\,\text{Strongly } n\text{-Mahlo} \mid n < \omega\}$	
	ZFC + $\forall n < \omega\, \exists!\,\text{Strongly } n\text{-Mahlo}$	
	ZFC + Weakly Compact	
	ZFC + Indescribable	B(ernays)
	ZFC + Subtle	
	ZFC + Almost Ineffable	
	ZFC + Ineffable	
	ZFC + $\{\exists!\, n\text{-Subtle} \mid n < \omega\}$	
	ZFC + $\forall n < \omega\, \exists!\, n\text{-Subtle}$	
	ZFC + $\kappa \to \omega$	
	ZFC + $\forall\alpha < \omega_1\ \kappa \to \alpha$	
	ZFC + $\exists!\, 0^\#$	
	ZFC + $\forall x \subseteq \omega\, \exists!\, x^\#$	
	ZFC + $\kappa \to \omega_1$	
	ZFC + Ramsey	
	ZFC + Measurable	

Table 5: Linear Hierarchy of Consistency Strengths, cont.

Higher-order arithmetics	1st-order set theories	2nd-order set theories
	ZFC + Concentrating Measurable ZFC + Strong ZFC + Woodin ZFC + Superstrong ZFC + Supercompact ZFC + Extendible ZFC + Vopenka ZFC + Almost Huge ZFC + Huge ZFC + Superhuge ZFC + $\forall n < \omega\ \exists!\, n$-huge ZFC + Rank into Itself ZFC + Rank + 1 into Itself NBG + V into V ZFC + Reinhardt ($\perp$)	

In thus paying homage to the formidable achievements of foundationalists in locating almost every theory of interest on the linear scale of consistency strengths, the present author nevetheless ventures to suggest that this achievement is a ladder that one is now free to kick away. If our main aim is to illuminate the structure of mathematical reasoning itself, then a natural logicist approach is preferable by far, rather than the straitjacketing that is involved in getting all these different theories into the forms indicated by the column-headings above. The natural logicist is prepared to treat each theory (and its objects) in a *sui generis* fashion, tailoring the formal rules for construction of proofs in a way that does analytic justice to the vast body of proofs in actual journal articles and textbooks written by, and for, the practitioners of these mathematical disciplines. Any foundationalist anxious to know the consistency strength of a theory T that holds one's interest can of course determine it by interpreting T in some T' (and interpreting T' in T), where T' is a theory that has already been calibrated by the methods we can now eschew. This is because (as Friedman has shown) sameness of consistency strength is equivalent to mutual interpretability.

References

[1] John P. Burgess. *Fixing Frege.* Princeton University Press, Princeton and Oxford, 2005.

[2] Harvey M. Friedman. Interpretations, According to Tarski'. Nineteenth Annual Tarski Lectures, *Interpretations of Set Theory in Discrete Mathematics and Informal Thinking.* Lecture 1. *Downloadable from http://www.math.osu.edu/ friedman/manuscripts.html*, 2007.

[3] Gerhard Gentzen. Untersuchungen über das logische Schliessen. *Mathematische Zeitschrift*, I, II:176–210, 405–431, 1934, 1935. Translated as 'Investigations into Logical Deduction', in *The Collected Papers of Gerhard Gentzen*, edited by M. E. Szabo, North-Holland, Amsterdam, 1969, pp. 68–131.

[4] P. Hajek and P. Pudlak. *Metamathematics of First-Order Arithmetic.* Perspectives in Mathematical Logic. Springer, Berlin, 1998.

[5] L. A. Harrington, M. D. Morley, A. Ščedrov, and S. G. Simpson. Introduction. In L. A. Harrington, M. D. Morley, A. Ščedrov, and S. G. Simpson, editors, *Harvey Friedman's Research on the Foundations of Mathematics*, pages vii–xii. North Holland, Amsterdam, 1985.

[6] Akihiro Kanamori. *The Higher Infinite. Large Cardinals in Set Theory from their Beginnings.* Perspectives in Mathematical Logic. Springer, Berlin, 1994.

[7] Dag Prawitz. *Natural Deduction: A Proof-Theoretical Study.* Almqvist & Wiksell, Stockholm, 1965.

[8] Peter Schroeder-Heister. A natural extension of natural deduction. *Journal of Symbolic Logic*, 49:1284–1300, 1984.

[9] Stephen G. Simpson. *Subsystems of Second order Arithmetic.* Perspectives in Mathematical Logic. Springer, Berlin, Heidelberg, New York, 1999.

[10] Craig Smoryński. Nonstandard models and related developments. In L. A. Harrington, M. D. Morley, A. Ščedrov, and S. G. Simpson, editors, *Harvey Friedman's Research on the Foundations of Mathematics*, pages 179–229. North Holland, Amsterdam, 1985.

[11] Alfred Tarski. The Concept of Truth in Formalized Languages. In J. H. Woodger, editor, *Logic, Semantics, Metamathematics*, pages 152–278. Clarendon Press, Oxford, 1956. First published, in Polish, in 1933.

[12] Neil Tennant. *Natural Logic.* Edinburgh University Press, 1978.

[13] Neil Tennant. Intentionality, syntactic structure and the evolution of language. In Christopher Hookway, editor, *Minds, Machines and Evolution*, pages 73–103. Cambridge University Press, Cambridge, 1984.

[14] Neil Tennant. *Anti-Realism and Logic: Truth as Eternal.* Clarendon Library of Logic and Philosophy, Oxford University Press, 1987.

[15] Neil Tennant. Conventional Necessity and the Contingency of Convention. *Dialectica*, 41:79–95, 1987.

[16] Neil Tennant. *Autologic.* Edinburgh University Press, Edinburgh, 1992.

[17] Neil Tennant. *The Taming of The True.* Oxford University Press, 1997.

[18] Neil Tennant. Ultimate normal forms for parallelized natural deductions, with applications to relevance and the deep isomorphism between natural deductions and sequent proofs. *Logic Journal of the IGPL*, 10(3):1–39, 2002.

[19] Neil Tennant. A general theory of abstraction operators. *The Philosophical Quarterly*, 54(214):105–133, 2004.

[20] Neil Tennant. Natural foundations for synthetic projective geometry. *Paper presented to the 8th Annual Midwest Philosophy of Mathematics Workshop at Notre Dame in October 2007*, 2008.

[21] Neil Tennant. Natural logicism via the logic of orderly pairing. In Sten Lindström, Erik Palmgren, Krister Segerberg, and Viggo Stoltenberg-Hansen, editors, *Logicism, Intuitionism, Formalism: What has become of them?*, pages 91–125. Synthese Library, Springer Verlag, 2008.

[22] Neil Tennant. Natural Logicist Foundations for Functions of Reals and their Derivatives. *Unpublished typescript*, 2008.

[23] Neil Tennant. The Schröder-Bernstein Theorem: A Case-Study for Natural Logicism. *Paper presented to the 10th Annual Midwest Philosophy of Mathematics Workshop at Notre Dame in October 2009*, 2009.

[24] Neil Tennant. Cut for Core Logic. *Review of Symbolic Logic*, forthcoming.

[25] Neil Tennant. Parts, Class, and *Parts of Classes*: An Anti-Realist Reading of Lewisian Mereology. *Synthese*, forthcoming.

[26] Neil Tennant. *What are the Reals, Really?* Unpublished typescript.