Naïve Mathematical Definition and Its Opposite

S. Blake Allan

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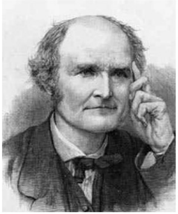
POMSIGMAA Special Session: Current Directions

Groups: Historical



Arthur Cayley (*On the Theory of Groups, as depending on...* $\theta^n = 1$, **1854**)

"A set of symbols, 1, α, β, ... all of them different, and such that the [associative] product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a *group*."

Felix Klein (*Erlangenprogramme Address*, **1872**)

"If now a given system of transformations has the property that any transformation obtained by combining any transformations of the system belongs to that system, it shall be called a *group of transformations*." [Footnote assumes inverses when $|G| = \infty$]

What does isomorphism mean?

Groups: Contemporary

"Pedestrian" Definition

A group is a set G, together with an operation $*$ on G which satisfies

- $g * h \in G$, (Closure)
- $g * (h * k) = (g * h) * k$, (Associativity)
- There exists an e such that $g * e = e * g = g$, (Identity)
- For each g, there exists a g^{-1} such that $g * g^{-1} = g^{-1} * g = e$ (Inverses)

"Structural" Definition

A group is any set G together with a relation $\cdot$ on G which satisfies

- $\forall x, y, z\big[x \cdot (y \cdot z) = (x \cdot y) \cdot z\big]$, (Associativity)
- $\exists e\big[\forall x[x \cdot e = e \cdot x = x] \wedge \forall x \exists y[x \cdot y = y \cdot x = e]\big]$, (Identity and Inverses)

up to isomorphisms of $(G, \cdot)$.

Groups: Structuralism

Saunders Mac Lane (*Some Recent Advances in Algebra*, 1939)

"Algebra tends to the study of the explicit structure of postulationally defined systems closed with respect to one or more rational operations."

Universality for Groups

Cayley's Theorem

Every group of order n is isomorphic to a subgroup of the symmetric group S_n.

Reformulation

Every abstract group acts on its labels!

Proof idea:

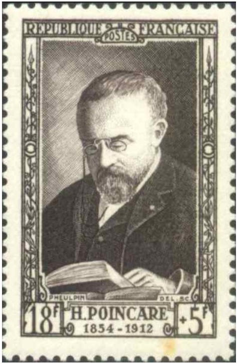
- AC implies G is in bijection with some ordinal α_G (say, via $\ell : G \longrightarrow \alpha_G$).
- Introduce an action of the group G on itself (e.g., $\Phi_g(h) = gh$ or $\widetilde{\Phi}_g(h) = ghg^{-1}$).
- Push this action through the bijection (e.g., $\Psi_g(\beta) = \ell\big(g\ell^{-1}(\beta)\big)$)
- This gives an isomorphism of G with some subgroup of S_{α_G}.

A Metaphysical Interlude

D. Hilbert & S. Cohn-Vossen (*Geometry and the Imagination*, 1932)

"...the tendency toward *abstraction* seeks to crystallize the *logical* relations inherent in the maze of material that is being studied...the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations."

A Metaphysical Interlude



Henri Poincaré (*Science and Method*, 1908)

"When the logician has resolved each demonstration into a host of elementary operations, all of them correct, he will not yet be in possession of the whole reality; that indefinable something that constitutes the unity of the demonstration will still escape him completely."

A Metaphysical Interlude

- *Theológos* (Metaphysics - objects apart from sensible things)
- *Mathematikós* (Mathematics)
- *Phusikós* (Physics - objects among corruptible things)

Axiomatic Definition

"Definition from above": specifying a set, structural postulates, and a notion of morphism.

Naïve Definition

"Definition from below": specifying a subobject of a universal setting, together with a context.

Measures

Definition

Let $\mathscr{B}$ be the Borel σ-algebra on $\mathbb{R}^n$. A function $\mu : \mathscr{B} \to [0, \infty]$ is a Borel measure if:

- $\mu(\emptyset) = 0$ (Normalization),
- For disjoint $(B_n) \subset \mathscr{B}$ with $\bigcup_{n\in\mathbb{N}} B_n \in \mathscr{B}$, we have $\mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sum_{n\in\mathbb{N}} \mu(B_n)$. (Countable Additivity).

Universality Theorem (Lebesgue Decomposition)

Any Borel measure μ on $\mathbb{R}^n$ has the canonical decomposition $\mu = \mu_{ac} + \mu_d + \mu_{sc}$, where $\mu_{ac} \ll \lambda^n$, μ_d is discrete, and $\mu_{sc} \perp \lambda^n$ but is not discrete.

Measures

Morphism

Let μ be a measure on X, and $f : X \to Y$ be measurable. Then $\nu(E) = \mu\big(f^{-1}(E)\big)$ defines a measure on Y, and we write $\nu = \mu \circ f^{-1}$ for the image of μ under f.

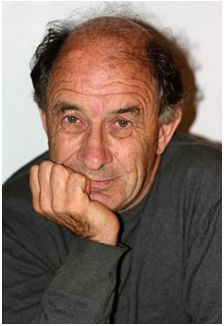
Properties

- A linear image of Lebesgue measure gives $\lambda^n\big(f(A)\big) = |\det(f)|\lambda^n(A)$ when $\lambda^n(A) < \infty$.
- Continuous images of measures preserve Radon-ness.

Theorem

a) If μ is atomless and probability ($\mu(X) = 1$), then there is a measurable function $f : X \to [0,1]$ such that $\mu \circ f^{-1} = \lambda$.
b) If also μ is Borel and positive on nonempty open sets, then it is homeomorphic to Lebesgue measure on $[0,1]$ iff X is homeomorphic to $[0,1] \setminus \mathbb{Q}$.

Interpretation



Vladimir Arnold (*Catastrophe Theory*, 1987)

"Abstract definitions arise in attempts to generalise 'naive' concepts while preserving their basic properties. Now that we know that these attempts do not lead to a real extension of the circle of objects (for manifolds, this was established by Whitney, for groups by Cayley, for algorithms by Church), would it not be better to go back to the 'naive' definitions in teaching as well?"

Conclusions

1) Naïve and axiomatic definitions account for the same mathematical objects, but in differently revealing ways.

2) Naïve definitions are to be preferred in pedagogical and heuristic discourse.

3) Axiomatic definitions are to be preferred in analytical/synthetical discourse.

Bonus

Henri Poincaré (*Science and Method*, 1908)

"One word more. The aim of each part of the statement of a definition is to distinguish the object to be defined from a class of other neighbouring objects. The definition will not be understood until you have shown not only the object defined, but the neighbouring objects from which it has to be distinguished, until you have made it possible to grasp the difference, and have added explicitly your reason for saying this or that in stating the definition."