Is school mathematics ``real" mathematics? Abstract # 1116-00-2437. Bonnie Gold, Monmouth University.

Abstract: One objection mathematicians often have to the work of philosophers of mathematics is that they don't ever discuss "real" mathematics. Many philosophers don't know mathematics beyond the school level (arithmetic and geometry); or if they know any advanced mathematics, it's often logic or set theory, which one can argue are somewhat anomalous. I've been interested for many years in trying to answer "what is mathematics?" by looking at topics that are on the boundary -- some would say they're mathematics, others would say they're not really mathematics. So I'd like to consider one such topic: is school mathematics "real mathematics" (as in David Corfield's *Towards a Philosophy of Real Mathematics*, or as what each of us means by the term)?

I'd like to thank the other officers of POMSIGMAA for inviting me to talk today; I'll say more about POMSIGMAA as the talk continues. Here's an outline of what I'll be saying: I'll give a bit of my own background, as well as of POMSIGMAA, then talk about the particular question I'm especially interested in and some approaches to the problem that have been considered so far. Then I'll discuss briefly what I'm trying to do to approach this problem, and turn to the particular aspect I'm considering today: school mathematics.

I. Where I'm coming from

I finished my Ph.D. in 1976. I knew a lot of mathematics (my thesis was in mathematical logic, but I loved point-set topology; algebra came naturally, though not category theory; analysis was not so natural), but was very influenced by a teacher of mine, Stan Tennenbaum, at the University of Rochester, to read Plato. So I started asking myself, how would I answer Socrates, if he asked me "What is mathematics?" I realized that all I could do is answer as many of his interlocutors did, with a list: "It's algebra, and geometry, and topology, and analysis, and differential equations, and logic, and number theory" and so on. But like Socrates' interlocutors, I didn't know how to find the answer that got at the essence of what mathematics is.

I started teaching at a small college, Wabash College, in the Midwest, and, once I got tenure and could work on whatever project interested me, I started turning more to the philosophy of mathematics. A leave of absence in 1984, accompanying my first husband to Belgium on sabbatical, allowed me to read work through the foundational period, up to the 1950s; and then a sabbatical of my own in 1991-92 allowed me to catch up on more recent work by a range of philosophers of mathematics: Philip Kitcher's *The Nature of Mathematical Knowledge*, Stephan Körner's *The Philosophy of Mathematics: An Introductory Essay*, Mark Steiner's *Mathematical Knowledge*, John Burgess' "Why I am not a nominalist", Penelope Maddy, Stewart Shapiro, Charles Chihara (though I was much more attracted to platonism than to Chihara's nominalist views).

I also read philosophical discussions of mathematics written by mathematicians – Davis and Hersh, for example. Three things stood out as I read work of the two communities: first, when mathematicians wrote about philosophy, they (we) were much more casual in their use of language than when they wrote about mathematics, whereas the philosophers were much more careful about the meanings of their words. Second, even for those philosophers who had some mathematical background, it tended to mostly be in logic or set theory, and so the examples used by philosophers tended to come mostly from either basic arithmetic, or from set theory, neither of which seem typical of mainstream mathematics. And the philosophers were not primarily addressing questions that really interested mathematicians – most mathematicians don't care that much about what kind of thing a number is: rather we care about how they interrelate, and about why an idea in one field often turns out to be useful in a field where there had previously seemed to be no connection. Third, the two communities were almost completely disjoint: they didn't appear to talk with (or possibly even be aware of) each other.

It seemed to me that *some* of the questions that philosophers were considering *were* of interest to mathematicians, and that it would be helpful to get some of the carefulness of thinking about philosophical questions into the philosophical writings of mathematicians. So I wrote "What is the Philosophy of Mathematics, and What Should it Be" (the title was a paraphrase of Dedekind's "Was Sind und Was Sollen die Zahlen?") to suggest that there were questions that are not being addressed by philosophers but that need to be; but also that some questions mathematicians are interested in *are* being discussed by philosophers, and to point to that literature.

I started reading more current work. Joe Auslander and I put together a panel for the joint winter meetings in 2001 on the philosophy of mathematics, with three mathematicians (Chandler Davis, Reuben Hersh, and Saunders Mac Lane) and two philosophers of mathematics (Kenneth Manders and Timothy Bays). I passed around a sign-up sheet for who would be interested in helping form a SIGMAA, and POMSIGMAA was born, approved by the MAA in 2002. I was the first Chair, Roger Simons was Chair-elect, Satish Bhatnagar was Program Director, Joe Auslander was Secretary, Charles Hampton was Newsletter editor/Webmaster, and Chuck Lindsey was Treasurer. POMSIGMAA has flourished, having contributed paper sessions at essentially every joint winter meeting since 2003, and having many major philosophers, and mathematicians interested in the philosophy of mathematics, give presentations at our meetings.

In 2003 and 2005 I gave talks at POMSIGMAA contributed paper sessions, first on what I think the question "What is mathematics?" should mean, and then on a possible direction of an answer. I've gotten involved, meanwhile, in all sorts of other projects having nothing to do with philosophy of mathematics. (I've been involved in assessment in undergraduate mathematics and in the mathematical education of teachers, served on the MAA's board of governors, started section versions of Project NExT in two MAA sections, and been very active in my MAA section), as well as in smaller issues within philosophy of mathematics. But now that I'm planning to retire at the end of this academic year, I'm hoping to finally come to an acceptable answer to the question I've been wondering about since I finished graduate school.

In recent years the parts of the philosophy of mathematics that interest me have flourished. (I'm not especially interested in foundational questions.) The Association for the Philosophy of Mathematical Practice was started in 2009 (with founding members Jean Paul van Bendegem, Jessica Carter, José Ferreirós, Marcus Giaquinto, Jeremy Gray, Abel Lassalle Casanave, Paolo Mancosu, Marco Panza, Jamie Tappenden). Issues in the philosophy of mathematical practice arose initially in work by Lakatos, and more recent work has come from many people, including Corfield's *Towards a Philosophy of Real Mathematics*. (Jessica Carter, one of its founding members, told me that my "What is the Philosophy of Mathematics" article had been influential in her development.) More recently, in 2012, Mic Detlefsen and others formed the Philosophy of Mathematics Association. Both organizations have gotten more communication going between philosophers and mathematicians interested in the philosophy of mathematics.

I think there are many interesting questions that are of interest to that group as a whole, and I'm now going to turn to the one that has been my own obsession.

II. Current approaches to "What is mathematics?", and (briefly) what I don't like about them

A. What are mathematical objects?

There are a lot of ways to interpret the question, "What is mathematics?" One natural interpretation is to turn it into the question, "What is the nature of mathematical objects?" This seems a natural approach, since mathematics is usually classified with the sciences, and that is how we approach questions such as "what is biology", "what is physics", "what is chemistry" – we look at the nature of the objects each studies. (from Webster's collegiate dictionary): Physics is the science that deals with matter and energy and their interactions; chemistry with composition, structure and properties of substances and of the transformations that they undergo; biology with living organisms and vital processes. But for mathematics, the same dictionary gives a list of areas of mathematics. Still, it would seem a reasonable approach that, if there were a good way to identify the nature of the objects of mathematical study in a coherent way that would be distinguishable from other types of objects people study, we could answer our question this way.

Most of the work by philosophers of mathematics on the question of what is mathematics has been in this direction. There has been a lot of work on the topic, primarily of the realist versus nominalist divide. If one is a realist (also called platonist), mathematical objects are mind-independent, time-and-space-independent abstract objects. Most mathematicians tend to think of mathematical objects this way – they're out there (somewhere) and we work with these objects with our minds – our minds don't change the objects, but we mentally poke and prod them to see how they behave. For philosophical realists, the challenge is how physical objects – human beings – can have access to these mind-independent objects – how we can perceive them, gain knowledge of them, even refer to them at all. More recent versions of this approach include structuralism, where mathematical objects are kinds of structures, and our knowledge of mathematical objects follows our knowledge of structures in the world. Nominalists deny the existence of mathematical objects, which takes care of that problem, but leaves the question of mathematical knowledge unaccounted for (sometimes they reduce it to logical knowledge), and certainly gives us no help whatsoever in describing mathematics in terms of the objects we study. Social constructivists (few philosophers classify themselves here, but a substantial number of mathematicians do) are somewhere between realists and nominalists. Mathematical objects are some kind of shared vision we have: one person defines an object and describes it well enough, to the rest of us, that we also can see it and work with it. There are also issues here, such as why humans, who can't seem to agree on virtually anything else – is there global warming? Is capitalism good or evil? – seem to be able to come to such uniform agreement about mathematical objects. But more importantly, from my perspective, I haven't seen an adequate description from the social constructivists about specifically <u>what</u> <u>kind</u> of socially-constructed objects mathematical objects are. Most of our objects of physics are also socially constructed, a lot of objects of society are socially constructed – which of these are mathematical objects (besides the circular "they're the socially-constructed objects mathematicians study").

I'm not sure that I have much to add to the debate among philosophers between nominalism and realism, and they're very busy working on this issue. But my main objection to this approach is that, even if we come down on the side of either realism or social constructivism, we've found the general realm of objects within which mathematical objects reside, but we don't have it narrowed down to the collection, or type of collection, of objects that are of interest specifically to mathematicians. New mathematical objects are discovered/described/developed/invented all the time: is there some way to describe what their general nature is (as distinct from a larger class)? If they're structures, which structures? Of course, we could work on this, and possibly the direction I'm going to suggest might lead to that kind of description in the end. However, for now it's not the main thrust of what I'm going to suggest.

B. What is mathematical methodology?

Notice that, while the various sciences are often defined based on the types of objects studied, science itself is not. Science tends more often to be defined in terms of its approach, how we acquire scientific knowledge. This is the second most common approach to describing what mathematics is. This approach proposes that mathematics is that which can be derived via logic. This approach also has a fairly long and honorable history: to a large extent, both the logicist and formalist foundational schools take this approach. For the logicists, mathematics is simply a subset of logic; they're sort of the realist side of this approach to defining mathematics. Formalism corresponds to the nominalist side: mathematical statements don't have meaning in themselves, and there's no inherent sense to the question of whether a mathematical statement is true or not: rather, can it be deduced from a given set of premises? But then, what is mathematics? Is it all formal deductions?

It is certainly true that, for our formal presentations of mathematics, most mathematicians believe that logical reasoning is central to our means of justification. But most mathematicians also know that, while one cannot work inconsistently, the logical structure is just a shell within which we work: as Bourbaki has written, it's just the "hygiene" of mathematics, that which keeps mathematics from becoming ill and corrupt. When we think about a mathematical problem, we are virtually never doing formal deductions: we're examining the concepts involved in the problem and how they fit together, how they behave, how they interact – somewhat like the descriptions of physics and chemistry given earlier. (For example, when considering how many real roots the polynomial $x^5 + x + 1$ has, we might first try to factor it but then we realize that since it's of odd degree, it's positive for large positive x, and negative for large negative x, and so it must have at least one root; and since its derivative is always positive, it's an increasing function, and so it has at most one root. Do you see any syllogisms being used here?) Some at the extremes (such as Zeilberger) suggest doing away with deductive mathematics entirely. Those slightly less extreme suggest that experimental mathematics will lead the way in the future. But for everyday working mathematicians, the logical or semi-logical deductions that we give when writing up our results for publication are only a shadow of the conceptual arguments that are how we got to those results, and the descriptions we give when we give a mathematics talk on the topic are somewhere in between. Lakatos was perhaps the first philosopher to describe that methodology in his *Proofs and Refutations*.

However, our actual methodology when we're discovering new mathematics, or even working on a standard college-level mathematical problem, is much wilder than logical deduction, and yet that kind of work has more of the flavor of real mathematics. There is work to be done describing this, and folks such as Corfield, in his *Towards a Philosophy of Real Mathematics*, is just beginning that work. In any case, for now we do not have a sufficient description of how mathematics proceeds that will let us use it as our answer to "What is mathematics?"

III. My proposed approach

So what I'm proposing to do is to look at the assorted subjects that make up modern mathematics, and ask what is common to all of them, *by virtue of which* we classify them as mathematics. And, along with this, for things that we decide not to classify as mathematics (but as mathematically-related fields, or fields that use mathematics), why don't we call them mathematics? I think this is a particularly good time in the history of mathematics to do this, because a range of fields of mathematics were developed during the 20th century (topology, algebraic geometry, mathematical logic, combinatorics, for example, each of which had roots in earlier mathematical questions or activities, but were primarily developed in the last century). At the same time, several fields that we would say either are not quite mathematics, or for which only some aspects are within mathematics, were also developed then, such as computer science and operations research. My suggestion is that we look at why we count some of these as mathematics and others as fields that use mathematics, but not as mathematics itself.

In particular, today I'm examining school mathematics. Obviously, in some sense we would call that the center of all mathematics. But there are other senses in which school mathematics doesn't quite make it. We all worry about the fact that students (especially if they haven't taken calculus in high school, but often even if they have) come to college thinking they want to major in mathematics, and then when they get here, and start taking our upper level courses, realize that this isn't the subject they thought it was. And yet, they've been studying "mathematics" for twelve or more years by then. So is school mathematics *real* mathematics? Can examining it help us think about what mathematics is, really?

IV. School mathematics So let's turn to school mathematics. I'm going to consider four different topics within this.

A. Traditional elementary-school arithmetic

So when I say "traditional elementary-school arithmetic", I mean what I learned in elementary school, which was largely arithmetic manipulations. When I was in school, back in the 1950s, we learned to count in kindergarten, to add in first grade, subtract in second grade, multiply in third grade, divide in fourth grade (I was fortunate enough to skip fourth grade; I learned long division on the first day of fifth grade), fractions in fifth grade, decimals and percents in sixth grade. (Seventh and eighth grades were review; algebra came in ninth grade.)

The emphasis was on the manipulations:

Compute 5364 ÷ 78 – here is most of it, in several steps:

```
     __6_
78 ) 5364
```

```
     __6_
78 ) 5364
    -468
```

```
     __68
78 ) 5364
   - 468_
      684
    - 624
```

To divide 5364 by 78, you observe that since 78 > 53, you must try to divide 536 by 78; that is, what is the largest single-digit multiple of 78 that is no larger than 536. You place that digit (which is a 6) above the 6 in 5364, multiply it by 78, place that result below the 536 and subtract it from 536. Then you bring down the 4, and repeat the process, putting the 8 above the 4. If you're at the stage of dealing with decimals, you might then put a decimal point to the right of the 4 in 5364, and some zeroes after it, and put a decimal point after the 8 in the quotient, bring down a 0 and continue for a while; if you're not at that stage, you'd subtract 8*78 and what is left is called the remainder. There was no discussion, or very little, of the meaning of all this manipulation. Since the only calculators at the time were about twice the mass of a typewriter (for you younger folks, that's a mechanical word processor in which the printer is combined with a keyboard, with the CPU being the human operating it), most people needed to do addition and subtraction by hand or mentally on a fairly regular basis, and many needed to be fairly proficient at multiplication and division – hence the year spent practicing each. We also learned methods to check our work, since it was important to consistently get the answers right, but there was very little, if any, explanation of why these algorithms worked, or what they meant.

Now, I'm not putting down this activity: in a time before widespread availability of electronic calculating machines (much less computers), there was a need for this. And it took humanity several thousand years to develop relatively efficient algorithms for these calculations – imagine trying to do the division problem I just mentioned in roman numerals.

But I would argue that this sort of activity is <u>not</u> mathematics, any more than predicting the coming of the messiah by giving numerical values to the various letters of the Hebrew alphabet and doing some computations with the Bible is doing mathematics. It's using mathematical developments for some human purpose, but it isn't doing mathematics. Why do I say this? What makes it seem not mathematics, to me, is that it lacks ideas, concepts, understandings, interrelations among meanings. In fact, slogans were developed to tell folks, "don't worry about any meaning: just do it" – "Yours is not to reason why: just invert and multiply" for division of fractions, for example. I don't think the word "mathematics" was used for this: we didn't have mathematics books; they were "arithmetic books." And I think those of us who are engaged in mathematics, whether professionally or even as amateurs, feel that meaning and understanding are central to doing mathematics.

Indeed, this view even trickled down to school teachers involved in teaching the "New Math" in the '60s, as exemplified by this quote from an eighth grade teacher [1, p. 470]: "Teaching the SMSG materials has made me realize that before using these texts, I was not teaching mathematics at all; I was simply teaching manipulative skills. Now I feel that I have been teaching at least some mathematics."

Not that I'm saying that inherently elementary-school arithmetic isn't mathematics – I'll come back to it in a little while – but I think that a certain approach to it isn't. A little while after I had moved to junior high school, the "New Math" came in. This movement began in many universities around the country; its aim was to put meaning into mathematics. The New Math began soon after the end of World War II, but became supported nationally and moved to the forefront after the launch of Sputnik in 1957. It tried to put meaning into school mathematics in some peculiar ways, partly motivated, I think, by the foundational difficulties mathematics itself had run into, and by the success of the Bourbaki-style formalist approach, together with axiomatic set theory. The proponents of the New Math proposed to put meaning into school mathematics by starting with a set theoretical and functions approach – counting consisted of one-to-one correspondences between sets, for example – as well as working in different number bases to give an understanding of arithmetic manipulations. But many proponents of the New Math also advocated a pedagogy of active learning, which I will discuss more later. Another eighth-grade teacher, teaching the New Math: "Most of the students of above-average ability reacted well to the material, obviously being glad to have something besides pages of problems on which to use their minds. *Some of the most striking cases of success, however, came from the ranks of students of low ability who made progress because they understood why they were performing certain operations.*" (italics added) A much more serious attempt was made with the New Math, than is currently being made with the Common Core State Standards, to bring teachers to the point that they were able to teach this material (there were many summer institutes for teachers, largely sponsored by the NSF). But its very alien, formal, from the intellectuals-to-the-masses French approach was totally rejected by the parents, leading to a "back-to-basics" movement – "if this is what meaning in mathematics is, let's get rid of it!"

B. Traditional high school mathematics

Much of high school mathematics has also been taught from a purely manipulative approach, and I would argue that it is then also only minimally mathematics. High school algebra is especially vulnerable to this approach. First, we introduce many new notations, often with little explanation of what they mean or the rules under which they operate, or even with ambiguous rules that change in what appears, to many students, arbitrary manners. For example, we use letters to represent assorted objects (generally at first, numbers, but later functions and more), and we do so in many different contexts. (We say that "$3(x + y) = 3x + 3y$" is an identity, the distributive property, whereas we sometimes say that "$\sin(x + y) = \sin x + \sin y$" is an error, misusing the distributive property – though it's often not explained why the distributive property isn't applicable here. However, the latter also could be asking us to find all x and y with that property. A lot of our notation is similarly ambiguous: the letter x can be a variable quantity in the definition of a function, an unknown quantity to be solved for, an arbitrary number in an expression of an identity, etc.).

Second, often very little emphasis is placed on the concepts involved: that to keep an equation true, or equivalent, what is done to one side must be done to the other; that if something doesn't work for numbers, it won't work for variables (but some things won't work for variables that do for numbers); that one manipulates expressions and equations to change them to forms where what's going on is more evident.

Of course, one doesn't have to teach algebra in this mindless way, but teachers who haven't made the connections, between the symbol-pushing and the concepts those symbols are representing, are unlikely to make them for their students.

So traditional high school mathematics is often taught in a way that is not mathematics – this is one reason I have pushed to not have "college algebra" count for general education credit in mathematics at the university level – because even if there are some concepts there, they tend to become overwhelmed by the manipulative activities.

On the other hand, the concepts are closer to the manipulations at the high school level than at the elementary school level, and some subjects, such as geometry, seem to me to almost inevitably require consideration of concepts. It can still be taught in a deadening way that makes it, at best, borderline mathematics. For many, the formality of two-column proofs, memorizing definitions, axioms and postulates, and so on, deadens the potential pleasure of discovering new mathematical results or justifications of observations. But inherently it seems much closer to "real" mathematics.

I've clearly, as I've been talking, been identifying some qualities that make something mathematics, or make it "not really mathematics." So maybe we're making some progress. Mathematics seems to be about ideas or concepts of a certain kind – among them, ideas that are preserved under some kinds of transformations. And mathematics seems to involve exploring these ideas, not just repeating or memorizing them. In particular, mathematics is not symbol manipulation devoid of a relationship to what those symbols represent. Symbols are important, for several reasons. We have learned that, once we know how that symbol manipulation affects the concepts, we can, in fact, manipulate the symbols without referring to their meaning. Thus learning how to manipulate symbols has been an important part of mathematical development. Further, in some circumstances, we can learn new relationships simply from manipulating those symbols (as, for example, when, in dealing with a differential equation that we don't know how to solve, we say, well, what if we had a solution in the following form: what would it then have to be). Nonetheless, the manipulations themselves are not mathematics: it's the concepts they are representing that is where the mathematics lies.

C. Common Core State Standards

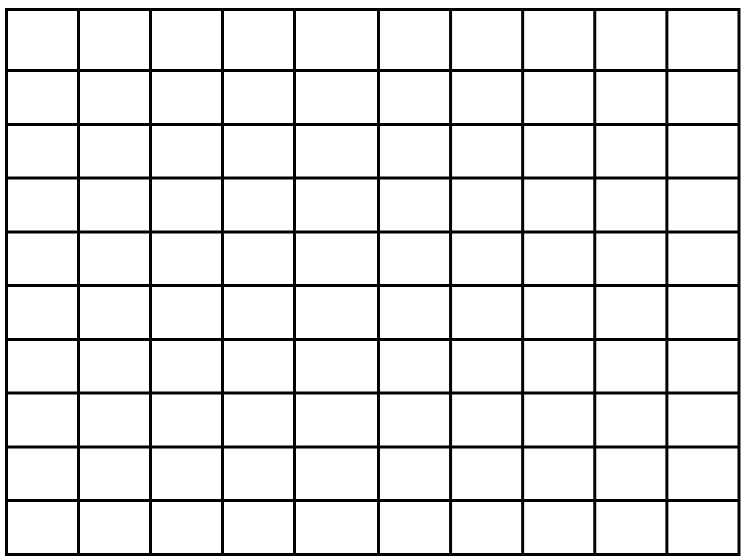
So I'd like to turn to a particular direction in school mathematics that has been receiving a lot of attention in the last half-dozen years, the Common Core State Standards. I'm not speaking either as a proponent or as an opponent of these standards, but more as an observer. There is clearly an attempt, via these standards, to get students thinking more about the meaning of the mathematics they are studying – we'll look at several examples in a few minutes. On the other hand, the standards have also been introduced in some very problematic ways and with a variety of agendas on the part of those introducing them. Unlike the New Math, not a lot of money has been spent on institutes training teachers to teach in ways consonant with the standards. They have been introduced at a time when many political leaders are looking for mechanisms to use to weed out ineffective teachers, or, alternatively, mechanisms to punish or blame teachers for the problems of society. I'm not interested in getting into a fight over this right now – clearly I do have some opinions about what I've seen, but that's not my point in bringing them up.

What I'd like to do is look at some of the sample assessment items for the CCSS that one of the consortiums developing those tests, PARCC, has released, and the extent to which these appear to be introducing real mathematics at the school level. I have chosen examples that I feel have some important mathematical concepts in them.

1. Third grade: "Art Teacher's Rectangular Array." This is classified as "modeling and applications", though, frankly, I think it's pretty poor as that. But it does involve exploration.

The question starts, "An art teacher will tile a section of the wall with painted tiles made by students in three art classes. Class A made 18 tiles, Class B made 14 tiles, Class C made 16 tiles."

"Part A: What is the total number of tiles being used?" (There's nothing interesting here: it's a straightforward sum. Notice it's 48 tiles in all.)



"Part B: The grid above shows how much wall space the art teacher can use. Use the grid to create a rectangular array showing how the art teacher might arrange the tiles on the wall. Select the boxes to shade them. Each tile should be shown by one shaded box."

Notice that 48 is actually quite an interesting number to think about in this context, since it has many factors, and thus there are many rectangular grids that one could make: 2 by 24, 3by 16, and so on. But this particular question is not written as a class exploration, though it being among the sample assessments suggests that a teacher getting children ready for these tests should have them do such explorations. For an assessment, they want a small finite number of possible student responses so that it will be fairly quick to grade.

Note that here the grid is 10 by 10; and students have to understand the terminology: that just shading the first 48 squares isn't what it's asking for. And although there are many factors of 48, to make it fit on this computer screen, only 6 by 8 or 8 by 6 will work. (Notice that a display that is 6 tiles wide and 8 high is quite a different display from one that is 8 by 6.) But asking students to explore, to find different rectangular arrays and see if there are some lengths that won't work, *is* doing real mathematics, exploring the factorization of integers.

2. Fifth grade: "Mr. Edmunds' Pencil Box" "Mr. Edmunds shared 12 pencils among his four sons as follows: Alan received 1/3 of the pencils; Bill received ¼ of the pencils; Carl received more than 1 pencil; David received more pencils than Carl."

"A. On the number line, represent the fraction of the total number of pencils that was given to both Alan and Bill combined." (This is an on-line problem, and students use buttons (labeled "More tick marks" "Fewer tick marks") to increase or decrease the number of equal sections on the number line.)

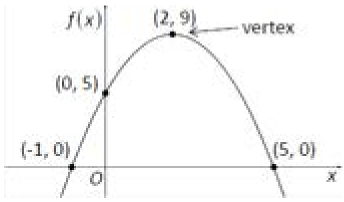
"B. What fraction of the total number of pencils did Carl and David **each** receive? Justify your answer."

The first part is entirely computational but can either be solved by simply adding 1/3 + 1/4, OR by first finding how many pencils Alan and Bill each received (by finding 1/3 of 12, ¼ of 12), adding the number of pencils, and then finding what fraction of 12 total pencils this was. To represent it on a number line, children have to understand the idea of denominators of fractions representing the number of parts a unit is divided into, and that therefore they need to make the screen have 12 parts (11 tick marks) between 0 and 1.

But the second part requires considerable reasoning: a student who found, in the first part, correctly, that Alan and Bill together got 7 pencils, knows that in total Carl and David got 5. But it requires further reasoning to realize that there is only one solution: since Carl gets at least two pencils, he must in fact get two and David must get 3, since he's supposed to get more. Again, because it's an assessment item, it's restricted to having a small number of correct answers (in this case, just one), but in class one could extend this problem either by not requiring that Carl gets more than one pencil (thus leaving numerous solutions), or by using a larger number of pencils and exploring what range of possibilities there are: it can be the beginning of a lot of combinatorial activity.

So we see several aspects of mathematics in these problems: the fact that there can be multiple approaches to solving mathematical problems, that they can be approached from several perspectives (numerical, geometric, via fractions or integers), that potentially open-ended problems can be used even in early grade school, that one can reason without having an algebraic (or other) formula, that one can do experiments with mathematical objects (even putting different number of tick-marks on the number line is a form of experiment). Let's turn to some high-school problems.

3. “High School Functions” The question says, “A portion of the graph of a quadratic function *f* (x) is shown in the *xy*-plane. Selected values of a linear function $g(x)$ are shown in the table.”



x	g(x)
-4	7
-1	1
2	-5
5	-11

For each comparison below, use the drop-down menu to select a symbol that correctly indicates the relationship between the first and the second quantity.		
First Quantity	**Comparison**	**Second Quantity**
The y-coordinate of the y-intercept f (x)	Choose from <, >, =	The y-coordinate of the y-intercept g(x)
f (3)	Choose from <, >, =	g(3)
Maximum value of f (x) on the interval -5 ≤ x ≤ 5	Choose from <, >, =	Maximum value of g(x) on the interval -5 ≤ x ≤ 5
$\frac{f(5) - f(2)}{5 - 2}$	Choose from <, >, =	$\frac{g(5) - g(2)}{5 - 2}$

Here we’re given two functions, neither via its algebraic expression: a quadratic function (with a negative coefficient of x^2) given by a graph, with the coordinates of all three intercepts and the vertex; and a decreasing linear function given by four successive points (with the *x*-coordinate changing at a fixed rate). This problem *can* be approached by finding the algebraic expressions for each function (though, to be honest, I’ve never done this because I find the other approaches more interesting). But in fact one can solve the whole problem without ever finding the algebraic expressions, by working entirely conceptually. For the first question, one is given the *y*-intercept of *f*, and simply by observing that *g* is decreasing and is below *f* 's *y*-intercept, one knows that *g*’s *y*-intercept is less than that of *f*. The second question is even quicker: *f* (3) is positive, while *g*(3) is clearly negative. The third question requires a little computation: because *g* is decreasing, its maximum on [-5,5] is at -5; and since its slope is -2, it will have the same maximum as *f* does. (One has to not be bothered by the fact that the maxima are at different *x*-coordinates for the two functions: this will probably bother many students who are used to comparing functions at the same *x*-value.) The fourth question is about the slope of what, in calculus, is called the “secant line”, of course; we have already found the slope for *g* in the third question, and just have to find it for *f*. It also involves comparing negative numbers (and, frankly, I don’t like all the 3’s hanging around).

So what do I see that involves what I’d call real mathematics in this problem? First, that the same object (the two functions) can be represented in multiple ways, and yet be the same object. Also, that these functions have properties that are characteristic of their type of object, the understanding of which lets one compare them in a range of ways. These properties are used for thinking about the concepts and determining further properties. So one explores the functions, via the ways they're presented; one "experiments" with them to see what they do (such as finding the slope of g and determining whether that slope is sufficient that it is higher than f somewhere on the given interval, whether its slope is always larger or smaller than that of f on that interval).

4. "Seeing structure in a quadratic equation" "Solve the following equation: $(3x - 2)^2 = 6x - 4$." This problem invites observing that this is a variation (translation and stretching) of the equation $u^2 = 2u$. Doing so enables the student to solve the equation very quickly and with much less symbolic computation or memorization, as well as via a much more conceptual approach. I think this is typical of "real" mathematics: not that we don't compute – even in this problem one has to do minor computation to see where $3x - 2$ equals 0 or 2. But I think we've all learned that it's generally a good idea to think about the problem for a while before starting to compute.

Possibly disturbing is that all of these problems can be approached both conceptually and via old-fashioned symbolic manipulation – but certainly seeing those connections is presumably the aim of those writing the problems, given the titles these problems have been given.

Problems such as the ones I have mentioned tend to encourage active learning: rather than memorizing something like the quadratic formula, one can ask students to look for structure in $(3x - 2)^2 = 6x - 4$: can they find some similarity in the expressions on each side of the equation? What would happen if we substituted u for $3x - 2$: what does that do to the equation? Can they then solve it? How would that help with the original equation? Similarly the high school function example: one can ask the class to think of properties of these functions that would allow answering the questions without finding formulas for the given functions. And elementary school students can experiment with ways to make a rectangle with area 48.

So what aspects of mathematics are we seeing in these CCSS problems? That the same object can be represented in multiple ways, and that by being aware of the multiple representations, one can often avoid nasty computations. This has been a productive attitude in mathematics, and, perhaps, is in fact part of the nature of mathematics. Also, that the excitement in mathematics is in problem-solving – an excitement that had largely been taken out of mathematics when I was in school – everything was rote or routine.

D. **Hyman Bass's school-level problems**

Finally, I'd like to look at something that Hyman Bass has been presenting recently – I heard him give a talk at the joint meetings in 2015, and he did a workshop (somewhat different presentation, but roughly the same topic) about it at the New Jersey section meeting. Bass is a well-known algebraist who has been interested in school level mathematics, and the mathematical education of teachers, for quite a number of years, and has worked with folks (especially Deborah Ball) with substantial experience actually teaching at the elementary school level. Actually, I'm first going to mention a problem I only heard him speak about at the MAA-NJ meeting: if you have *m* cakes to divide among *n* people (say 5 cakes among 7 people), what is the smallest number of cuts you can make to divide them evenly? This is clearly a problem 3rd grade students can work on; fair sharing is of interest at that age, and they have the sophistication to work on it. There are many ways to cut the cakes: cut each cake 5/7 of the way; then, with one of the 2/7 pieces, cut it in half. So 5 people get uncut 5/7 pieces, and the remaining two get three pieces: two 2/7 pieces and one 1/7 piece. Or, cut the first cake at 5/7; put the remaining 2/7 together with 3/7 of next cake. Next, take its remaining 4/7 together with 1/7 of next cake. This is the only cake cut twice: at 6/7 also. And so on.

But it turns out that the question is related to the Euclidean algorithm, to square tilings of a 5 by 7 rectangle, to several questions in graph theory, and so on.

But the main thing I've seen him do is hand out a list of four arithmetic problems to one group, three rate problems to another, three geometry problems to a third group, three algebra problems to a fourth group, and suggest that they work on them for a while. Here, for example, are the four arithmetic problems:

1. Find all ways to express ½ as the sum of two unit fractions

2. Find all rectangles with integer side lengths whose area and perimeter are numerically equal.

3. The product of two integers is positive and twice their sum. What could these integers be?

4. For which integers $n > 1$ does $n – 2$ divide $2n$?

For contrast, here is one of the geometry problems: Given a point *P* in the plane, find all integers *n* such that a small circular disk centered at *P* can be covered by non-overlapping congruent tiles shaped like regular n-gons that have *P* as a common vertex.

And one of the rates problem: Nan can paint a house in *n* days, and her mom can paint it in *m* days (*n* and *m* positive integers). Working together, they can paint the house in 2 days. What are the possible values of *n* and *m*?

First, observe that these are all school-level problems – that is, they could be posed to high school classes, or lower. Of course, all mathematicians know that there are very difficult problems that can be posed with high school, or even elementary school, vocabulary.

Also notice that they all involve some symbol manipulation, but also involve understanding and examining some concepts, and what the implications of the concepts are. One fairly quickly sees that the first problem has two solutions: ¼ + ¼, or 1/3 + 1/6. One doesn't see any more: but how can one show that that's all there are? Hmm: $1/m + 1/n = (m+n)/mn$. So if $1/m + 1/n = ½$, $2(m+n) = mn$ – wow – perimeter equals area; the product is twice the sum...

In any case, as people work on their collection of problems, they start realizing these are all variations of the same problem. That is, the same question can be seen through over a dozen different lenses. I think that this collection of problems embodies at least a significant aspect of the nature of mathematics – but as of today, I'm still puzzling over them, trying to figure out exactly what that is.

Notice that both the CCSS problems I mentioned and those Bass has been presenting have the feature that they can be approached not simply by different methods, but by methods that are from completely different parts of mathematics – geometry, algebra, graph theory, and so on. Certainly there are theorems in mathematics that have never been approached from the perspective of more than one area of mathematics. But I think the frequency with which one can approach a mathematical concept from a range of perspectives is a feature that intrigues people about mathematics, that makes it an exciting subject, and perhaps that makes it different from other subjects. Not that this doesn't happen in other subjects, but these deep, unexpected connections occur with an unusually high frequency in mathematics.

One thing I'd like to point out about this kind of problem: it makes any "foundation" for mathematics that I've seen – whether through logic, set theory, model theory, or category theory – inadequate. None of them account for the wide range of different pairings of fields and approaches one finds in mathematics. Logic and model theory are especially susceptible to this criticism – simply changing whether you're describing the natural numbers with just one constant (1) and one function (the successor) versus with several functions (say, addition and multiplication) makes it, from the perspective of logic, completely different structures. One can interpret one within the other – there are ways of partially accounting for this difficulty – but this is only for perspectives that are from the same field of mathematics. Logic really seems inadequate to view mathematical objects from different perspectives. But so does category theory – it does enable one to talk about certain ways in which very different structures are essentially the same – but only when there are clear ways of replacing every item in one perspective with its equivalent from the other. It's one reason that I, and many mathematicians, tend to be, at least in some respects, firm platonists. If it's a world out there that we're reporting on, then mathematicians may well be somewhat like the blind men examining the elephant – we each perceive it from our own perspective, and are amazed when someone shows us how to see it from a different perspective.

V. Summary

One problem with working on the question "What is mathematics?" is that, if I find the right answer, it will be obvious to people, and the reaction will be, "Yes, of course, I knew that all along." Nonetheless, I think it's worth trying to find it.

So let me try to summarize where this has taken us.

First, I've been trying to look at the question of what mathematics is by examining what is in common about all the subjects that we call mathematics that is behind our calling them mathematics, and for subjects that are on the borderline, or are mathematically-related, why we would not call them mathematics.

For today, I'm examining school (K-12) mathematics. As I've suggested, I'd classify some school mathematics – what I'd call "mindless symbol manipulation" – as not really being mathematics, but I'd say that, depending on what is taught and how it is taught, and how questions are asked, much of what is taught at the K-12 level can be mathematics; and that there is currently an attempt, in the CCSS, to bring more real mathematics into the curriculum.

In particular, what aspects do we see of the nature of mathematics as we examined school mathematics? First, that the same object can be represented in multiple ways, and that by being aware of the multiple representations, one can often avoid nasty computations. That the same structure may occur in completely different contexts and even completely different branches of mathematics. That mathematical understanding and knowledge comes from working on problems that seem natural, by experimenting with what can happen and what is ruled out by the given conditions. That mathematics is about concepts, understandings, interrelations among meanings: there needs to be an interaction between the concepts and mathematical manipulations to solve mathematical problems. We experiment and make conjectures; then we look for proofs, which often involves more experimentation as well as symbolic computation.

None of this is news; but we're focusing on what makes mathematics mathematics – so maybe we're making a little progress.

I would welcome any comments or suggestions.

Reference:

[1] American Association for the Advancement of Science (AAAS)Archives. Box 222/B-1-4.John R. Mayor Office Files, 1961-1966. File: "Correspondence – Information and Reports – 1961." American Association for the Advancement of Science Headquarters, Washington, D.C., quoted in "The Original New Math: Storytelling versus History," by David L. Roberts and Angela L. E. Walmsley, *The Mathematics Teacher*, Vol. 96, No. 7 (October 2003), pp. 468-473.