Philosophical implications of the paradigm shift in model theory

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MATHFEST

Papers and lecture slides with much of this are on my website.

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Forthcoming book

Model Theory and the Philosophy of Mathematical Practice:

Formalization without Foundationalism

Three perhaps unfamiliar phrases

1. Model Theory
2. Philosophy of Mathematical Practice
3. Formalization without Foundationalism

A model theorist is a

SELF CONSCIOUS MATHEMATICIAN

We speak about structures, which might be groups, linear orders, differentially closed fields etc.

And formal theories about these structures.

To explain the paradigm shift we will introduce some basic model theoretic concepts.

Association for the Philosophy of Mathematical Practice

Goals include

Foster the philosophy of mathematical practice, that is, a broad outward-looking approach to the philosophy of mathematics which engages with mathematics in practice (including issues in history of mathematics, the applications of mathematics, cognitive science, etc.).

`http://www.philmathpractice.org/about/`

Midwest PhilMath Workshop 18 (MWPMW 18) Notre Dame October 14/15, 2017 `https://philevents.org/event/show/33270`

Mathematical and Philosophical Impact

Model theoretic formalization is a powerful tool for organizing and doing mathematics and for the philosophy of mathematical practice.

Formalization without Foundationalism

Bourbaki

Dieudonne  Bourbaki  Cartan 

Bourbaki distinguishes between ‘logical formalism’ and the ‘axiomatic method’.

‘We emphasize that it (logical formalism) is but one aspect of this (the axiomatic) method, indeed the least interesting one’.

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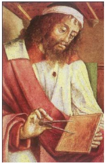
We reverse this aphorism:
The axiomatic method is but one aspect of logical formalism.

And the foundational aspect of the axiomatic method is the least important for mathematical practice.

Euclid-Hilbert formalization 1900:

Euclid  Hilbert 

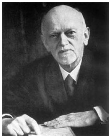
The Euclid-Hilbert (the Hilbert of the Grundlagen) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.

But the arguments and statements take place in natural language.

For Euclid-Hilbert logic is a means of proof.

I could add Bourbaki to the title.

Hilbert-Gödel-Tarski-Vaught formalization 1917-1956:


Hilbert


Gödel


Tarski


Vaught

In the Hilbert-Gödel-Tarski-Vaught framework, logic is a mathematical subject.

This Hilbert is the founder of proof theory.
Vocabulary is chosen for the particular topic.
Explicit rules define a formal language and proof.
Semantics is defined set-theoretically.

The completeness theorem establishes the equivalence between syntactic and semantic consequence.

Formalization of a mathematical area

Definition

A *full formalization* involves the following components.

1. Vocabulary: specification of primitive notions.
2. Logic
    1. Specify a class of well formed formulas.
    2. Specify truth of a formula from this class in a structure.
    3. Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

This talk focuses on first order logic.

Vocabulary, structures, truth

Specify the most basic notions of a particular area.

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Vocabulary, structures, truth

Specify the most basic notions of a particular area.

- A vocabulary L is a collection of relation and function symbols. e.g. $+, \cdot, 0, 1$
- A structure for that vocabulary (L-structure) is a set with an interpretation for each of those symbols.

  $(\mathbb{N}, +, \cdot, 0, 1)$ is the structure of the natural numbers.
  $(\Re, +, \cdot, 0, 1)$ is another structure for the same vocabulary.

Logic

- The first order logic ($L_{\omega,\omega}$) associated with the vocabulary L is the least set of formulas containing
  1. the atomic L-formulas
  2. closed under **finite** Boolean operations and
  3. quantification over finitely many individuals.

Examples:

atomic formula: $y = x + 7$
(defines a line in e.g. $\Re^2$)

Crucial notion

A definable set is the set of solutions in M^n of a formula $\phi(\mathbf{x}, \mathbf{m})$.

Sentences, Axioms, and Theories

quantification $(\exists x)\ x^2 = 2$

- Sentences are formulas that are either true or false in a structure
  $\Re \models (\exists x)\ x^2 = 2$
  $\mathbb{N} \models \neg(\exists x)\ x^2 = 2$

A theory T is a collection of L-sentences. Contemporary model theory focuses on theories not logics.

Theories specify the particular area being studied

They can be given by explicit axioms (groups, first order Peano)
or as $\mathrm{Th}(M)$,
e.g. $\mathrm{Th}(\mathbb{N})$ is true arithmetic.

Formalizing an area: Algebraic Geometry

Webster

a branch of mathematics concerned with: the study of sets of points in space of n dimensions that satisfy systems of polynomial equations in which each equation contains n variables

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Model Theory (half true)

The study of definable subsets of algebraically fields.
i.e. models of the complete theory ACF_p (p varies)

Why the same

Weil's universal domains are 'saturated' models of the theory ACF_p.
Tarski/Robinson proved: Definable subsets of algebraically closed fields are boolean combinations of equations.

First order Model Theory before 1960:

Fundamentals

1. **Theorem** [Löwenheim-Skolem]. If a first order theory has an infinite model, it has a model in each infinite cardinality.
2. **Theorem** [Compactness]. If every finite subset of a collection Σ of sentences has a model then Σ has a model.
3. **Theorem** [Completeness].
   A sentence ϕ is deducible from a theory T ($T \vdash \phi$)
   iff
   if $M \models T$ then $M \models \phi$

Note: Grothendieck

Foundationalism - the search for reliability

Spurred by the paradoxes, in the first half of the twentieth century logicians used formal theories to study the *certainty* of mathematics.

Higher order logic was the main tool until the 30's. Gradually, the foundational tool became set theory.

The first order theory of sets: ZFC has had immense success in understanding fundamental concepts.

This study is largely disjoint from the rest of modern mathematics.

Because, it seeks a common foundation for all of mathematics.

We seek 'local' foundations to preserve the ethos of each area.

Complete Theories

Definition

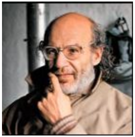
T is complete if for every ϕ, either ϕ or $\neg\phi$ is in T.

Examples

A complete theory can be

1. given as axioms: Algebraically closed fields (of fixed characteristic), dense linear order, differentially closed fields
2. or as sentences true in a single (or class of structures)
   (a) $\mathrm{Th}(\mathcal{C}, +, \cdot, 0, a), \mathrm{Th}(\mathbb{Q}, <)$,
   (b) Theory of free non-abelian groups: any two nonabelian free groups have the same first order theory.
   (2006) Sela / Kharlampovich & Myasnikov (priority ??)

Use of formalization: Ax-Grothendieck:

Ax  Grothendieck 

Theorem: 1968, 1966

Every injective polynomial map on an affine algebraic variety over $\mathcal{C}$ is surjective.

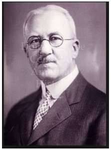
The Ax model theoretic proof:

$1-1$ implies onto is axiomatized by $\forall\exists$ - sentences:
for every polynomial function f and *every* possible value b *there is* an a with $f(a) = b$

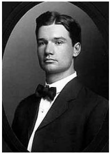
So preserved from finite fields to the algebraically closed $\tilde{F}_p$
The axioms of the complete theory ACF_0 show any sentence true in almost all finite characteristics is true in $\mathcal{C}$.

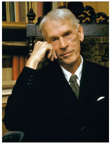
Categoricity

American Postulate Theorists:


E.Huntington


E.H. Moore


R.L. Moore


O. Veblen

A COMPLETE SET OF POSTULATES FOR THE THEORY OF ABSOLUTE CONTINUOUS MAGNITUDE* (PROC AMS 1902)
BY EDWARD V. HUNTINGTON

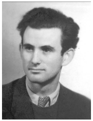
“The following paper presents a complete set of postulates or primitive propositions from which the mathematical theory of absolute continuous magnitude can be deduced.”

3 intertwined notions

The distinction between

1. semantic completeness: $T \vdash \phi$ iff $T \models \phi$
2. categoricity: T has only one model.
3. deductive completeness: For every ϕ, $T \vdash \phi$ or $T \vdash \neg\phi$

was not really understood until the 1930's.

Categoricity in Power 1954: Łoś 

A first order theory

T is categorical in power κ if it has exactly one model in cardinality κ.

Łoś conjecture, Morley's theorem

If a countable theory is categorical in one uncountable power it is categorical in all uncountable theories.

Examples

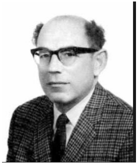
- Algebraically closed field of fixed characteristic.
- vector spaces over a fixed field
- torsion free divisible abelian groups

Our Argument

1. Categoricity in power implies strong structural properties of each categorical structure.
2. These structural properties can be generalized to all models of certain (syntactically described) complete first order theories.

Virtuous Properties and The Paradigm Shift

The Significance of Classes of Theories : Definability

Tarski  Robinson 

Quantifier Elimination and Model Completeness

Every definable formula is equivalent to quantifier-free (resp. existential) formula.

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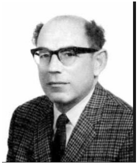
Tarski  Robinson 

Quantifier Elimination and Model Completeness

Every definable formula is equivalent to quantifier-free (resp. existential) formula.

Tarski proved quantifier elimination of the reals in 1931.

Such a condition provides a general format for Nullstellensatz-like theorems.

Robinson provides a unified treatment of Hilbert's Nullstellensatz and the Artin-Schreier theorem which led to the notion of differentially closed fields.

Quantifier-elimination provides the epistemelogical virtue of accessiblity.

pragmatic criterion

Properties of theories: complete, model complete, decidable, categorical, categorical in power, ω-stable, stable, $\pi_2 - axiomatizable$, finitely axiomatizable

Criterion

A property of a theory T is virtuous if it has significant mathematical consequences for T or its models.

Under this criteria

1. 'elimination of quantifiers' is virtuous.
2. completeness of a first order theory is virtuous.
3. categoricity in uncountable power of a first theory (with infinite models) is virtuous.

Complete Theories Kahzdan 

Complete theories are the main object of study.
Kazhdan (in intro to his notes on Motivic Integration):

> *On the other hand, the Model theory is concentrated on [the] gap between an abstract definition and a concrete construction. Let T be a complete theory. On the first glance one should*
> *not distinguish between different models of T, since all the results which are true in one model of T are true in any other model.*
> *One of the main observations of the Model theory says that our decision to ignore the existence of differences between models is too hasty.*
> *Different models of complete theories are of different flavors and support different intuitions.*

Describing the relation of points and models

Definition

The collection of formulas p is a complete type over A if it satisfies one of the following equivalent conditions.

1. p is a maximal consistent set of formulas $\phi(x, \mathbf{a})$ with parameters $\mathbf{a}$ from A.
2. The solutions of p are an orbit of the group of automorphisms of the monster model (universal domain) which fix A.

$S(A)$ denotes the set of such p.

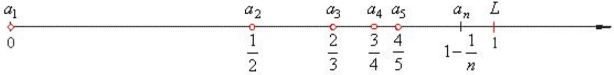
Thus $S(A)$ is the collection of descriptions of 1-pt extensions of A.

Understanding Types

The theory of an dense linear order without end points with an infinite increasing sequence named.

(3) $a_n = 1 - \frac{1}{n}$ for $n = 1, 2, 3, \ldots$ gives the sequence, $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

shown on the number line



Question to audience

Let $\mathcal{A} = (\mathbb{Q}, <, a_n)_{n<\omega}$ be the rational numbers and let a_n denote $1 - 1/n$.

Do 1 and 1.1 realize different types over $\{a_n : n < \omega\}$?

Different flavors of models

Are $\mathcal{A}_{sat} = (\mathbb{Q}, <, a_n)_{n<\omega}$ and $\mathcal{A}_{ord} = (\mathbb{Q} - \{1\}, <, a_n)_{n<\omega}$ isomorphic?

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Note that $\mathcal{A}_{prime} = ((-\infty, 1), <, a_n)$ is the third countable model of T.

Some favorite flavors

prime: Each realized type is generated by a single formula

saturated: all n-types realized for all n

What paradigm shift?

Before

The paradigm around 1950 concerned the study of logics; the principal results were completeness, compactness, interpolation and joint consistency theorems.

Various semantic properties of theories were given syntactic characterizations but there was no notion of partitioning all theories by a family of properties.

What paradigm shift?

After

After the paradigm shift there is a systematic search for a finite set of syntactic conditions which divide first order theories into disjoint classes such that models of different theories in the same class have similar mathematical properties.

In this framework one can compare different areas of mathematics by checking where theories formalizing them lie in the classification.

What is the role of Logic?

Logic is the analysis of methods of reasoning

versus

Logic is a tool for doing mathematics.

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Logic is the analysis of methods of reasoning

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Logic is a tool for doing mathematics.

More precisely,
Mathematical logic is tool for solving not only its own problems but for organizing and doing traditional mathematics.

Counting types

Fix a complete theory T

Notation

A complete n-type over a set A is a description of an n-tuple (over the empty set).
$S(A)$ is the collection of types over A

Definition

The complete theory T is λ-stable if for every $M \models T$ and every $A \subset M$,

$$|A| \leq \lambda \Rightarrow S(A) \leq \lambda.$$

This classification using arbitrary large cardinalities is reflected by mathematically significant properties of small models.

Semantic classification of first order theories

Theorem

Every countable complete first order theory lies in exactly one of the following classes.

1. (unstable) T is stable in no λ.
   NO STRUCTURE THEORY
2. (strictly stable) T is stable in exactly those λ such that $\lambda^\omega = \lambda$
   LOCAL DIMENSION
3. (superstable) T is stable in those $\lambda \geq 2^{\aleph_0}$.
   BETTER CONTROL
4. (ω-stable) T is stable in all infinite λ.
   ALMOST ALGEBRAIC GEOMETRY

Stability is Syntactic

Definition

T is unstable if just if some formula linearly orders an infinite subset of each model of T.

This formula changes from theory to theory.

1. dense linear order: $x < y$;
2. real closed field: $(\exists z)(x + z^2 = y)$,
3. $(\mathbb{Z}, +, 0, \times)$:$(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
4. infinite boolean algebras: $x \neq y$ & $(x \wedge y) = x$.

Why does this matter to mathematicians?

1. (unstable)
   linear order, Boolean algebras, set theory, Peano Arithmetic
2. (strictly stable)
   separably closed fields, $(Z, +, 1)^{\omega}$, DCF_p, free non-abelian groups, any abelian group
   $(\mathcal{C}, +, , G)$ where G is the finitely generated group from Mordell-Weil conjecture
3. superstable
   $(Z, +, 1)$, (Z_p^{ω}, H_i), finitely refining sequences of equivalence relations
4. (ω-stable)
   ACF_0, ACF_p, matrix rings over ω-stable fields, $((Z_4)^{\omega}, +)$, DCF_0, complex compact manifolds,

Shelah classification strategy 

A property P is a dividing line if
both P and $\neg P$ are virtuous.

Stable and superstable are dividing lines

ω-stable and $\aleph_1$-categorical are virtuous but not dividing lines.

Geometry

Geometry

1. is the source of the idea of axiomatization and
2. through the medium of geometric stability theory plays a fundamental role in analyzing the models of tame theories *tame mathematics*

Dimension: the essence of geometry

Dimension is a natural generalization of the notion of two and three dimensional space.

If a geometry is coordinatized by a field the dimension tells us how many coordinates are needed to specify a point.

Zilber, Hrushovski, Pillay, Buechler, Newelski, Chatzidakis, Bouscaren, Poizat

Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \ X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $\mathrm{cl}(x) = x$ the structure is called a geometry.

STRONGLY MINIMAL

Definition

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Definition

a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

$\aleph_1$-categorical theories

Morley  **Lachlan**  **Zilber** 

Theorem

A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

These two conditions assign a unique dimension which determines each model of T.

Strongly minimal sets are the building blocks of structures whose first order theories are categorical in uncountable power.

$\aleph_1$-categorical theories

Definition

A model M of a complete theory T is prime over a subset X if every morphism from X into a model N of T extends to a morphism of M into N.

Theorem (Baldwin-Lachlan)

If T is categorical in some uncountable power, there is a definable strongly minimal set D such that every model M of T is prime of over $D(M)$.

Thus, the dimension of $D(M)$ determines the isomorphism type of M.

The role of geometry

If T is a stable theory then there is a notion ‘non-forking independence’ which has major properties of an independence notion in the sense of van den Waerden.

It imposes a dimension on the realizations of regular types.

For many models of appropriate stable theories it assigns a dimension to the model.

This is the key to being able to describe structures.

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Bourbaki's 3 great mother structures

order, groups, topology

ADD geometry

Geometric Stability Theory

Classification

The geometries of strongly minimal sets fall into 4 classes:

1. discrete (trivial) $(\mathrm{cl}(ab) = \mathrm{cl}(a) \cup cl(b))$
2. modular or vector space like: (the lattice of closed subsets of the geometry is a modular lattice).
3. field-like (somehow bi-interpretable with a field).
4. none of the above: non-desarguesian but not vector space like.

This classification has immense consequences in both pure and applied model theory.

Geometry and Algebra are inevitable

Hrushovski 

Zilber / Hrushovski

Abstract model theoretic conditions imply algebraic consequences.

e.g. A group is definable in any $\aleph_1$-categorical theory that is not almost strongly minimal.

More technical hypothesis imply

1. the group is an abelian or a matrix group over an ACF of rank at most 3 or
2. there is a definable field.

The hypothesis do not mention anything algebraic.

Why does this matter?

Why does this matter to model theorists and philosophers?

The Main Gap: No Structure or structure

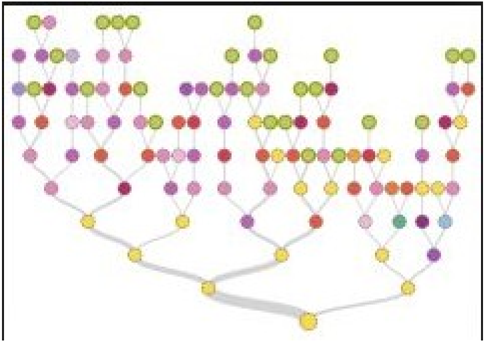
Let T be a countable complete first order theory. Either,

1. The countable models have almost all the information.

   Each model of cardinality λ is decomposed into countable models indexed by a *tree* of countable height and width λ.

   or
2. T has the maximal number of models in every cardinality and there is no uniform scheme of assigning countable invariants.

Decomposability



Arbitrary width; but only countably high.
The colors represent various dimensions.

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Does this matter to mathematicians?

First order analysis

1. Axiomatic analysis: (differentially closed fields, transseries, and surreal numbers)

   Models are fields of functions:

   Solves problems dating back to Painlevè 1900
   Applications to Hardy Fields, and asymptotic analysis

   Aschenbrenner, V.d. Dries, V.d.Hoeven, Freitag, Moosa, Pillay, Scanlon, . . .

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2. Definable analysis (*o*-minimality)

   Functions are defined explicitly :
   real exponentiation, number theory

   Wilkie, Pila, Peterzil,Starchenko, Marker, Macintyre, ...

Does this matter to mathematicians?

Substantial Applications

1. number theory and Diophantine geometry
2. real algebraic geometry
3. compact convex manifolds
4. real exponentiation
5. complex exponentiation
6. differential algebra
7. motivic integration
8. asymptotic analysis
9. combinatorial graph theory

Current internal developments: neo-stability

Cernikov, Boney, Conant, Malliaris, Simon, Terry, Vasey

The wild world of mathematics

Point

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The wild world of mathematics

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Counterpoint

We can systematically make this separation in important cases. Gödel showed us that the wild infinite could not really be separated from the tame mathematical world if we insist on starting with the wild worlds of arithmetic or set theory.

Summary

The crucial contrast is between a foundationalist approach – a demand for global foundations
and a foundational approach – a search for mathematically important foundations of different topics.

1. Formalization is a potent tool to do mathematics.
2. The classification of a first order theories sharpens this tool.
3. It is also a tool for philosophers of mathematics.

Summary

1. Contemporary model theory makes formalization of *specific mathematical areas* a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).
2. Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

Two Further theses

3. The choice of vocabulary and logic appropriate to the particular topic are central to the success of a formalization. The technical developments of first order logic have been more important in other areas of modern mathematics than such developments for other logics.
4. The study of geometry is not only the source of the idea of axiomatization and many of the fundamental concepts of model theory, but geometry itself plays a fundamental role in analyzing the models of tame theories.