Strict Finite Foundations of Mathematics

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What is Strict Finitism?

- No general account of the philosophical stance.
- It is an anti-realist position with respect to mathematics.
- It falls under a board understanding of Constructivism, but takes many ideas to the extreme.

A Definition?

- There are finitely many natural numbers.
- Mathematics should only be concerned with objects or concepts that are accessible by constructions or procedures that can be executed or performed by methods available to an actual human being.

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Distinction from Constructivism/Intuitionism

- Replaces constructible/possible in principle with constructible/possible in practice.

Origins of Strict Finitism:

- Only discussing academic published work after 1900.
- Yesenin-Volpin: Ultra-Intuitonism
  - This work is extremely cryptic.
  - specific purpose, finitary consistency proof of ZFC Set Theory.
- Wittgenstein: Remarks on the Foundations of Mathematics
  - discusses feasibility and what is mathematically executable by real humans
  - Does not contain a systematic proposal.

A Better Origin Story:

- Van Dantzig "Is $10^{10^{10}}$ a finite number?" (1956)
- Isles "What Evidence is There That 2^{65536} is a Natural Number?" (1992)
  - Can you represent such numbers in Arabic Numerals?
  - "[2^{65536}] represents a number which exceeds the total number of vibrations executed by all subatomic particles of size $< 10^{-30}$ cm (smaller than a quark!) which would be needed to fill a universe of radius 10^{12} light years (larger than the observational diameter of the universe!) were each vibrate 10^{50} times per second over a period of 10^{12} years (longer than the surmised age of the universe!)."
  - These 'numbers' are not feasible.
  - Exponentiation is not a total function on the natural numbers. (Induction is self-referencing: Impredicative)
  - Proposed alternatives suffer from poverty.

Rich Finitism: work of Priest and Van Bendegem

- Inconsistent Mathematics:
  - Priest has devised a finite, axiomatic, complete, **inconsistent** model of arithmetic.
  - All true statements in Peano arithmetic are true of the model.
  - There are no inconsistencies pertaining to statements about numbers below a certain threshold.

Operationalism:

- Bridgeman: The Logic of Modern Physics, 1927

Geometric Constructions:

- Geometric constructions can be interpreted as physical constructions with instruments.
- The formal relations can have interpretations such that their truth values can be determined by executable operations with physical instruments.

Strict Finite Systems Outside of Arithmetic

Physical Operationalism: Geometric Constructions:

A Strict Finite Foundation for Geometric Constructions, Axiomathes, 2022

- A first-order, quantifier-free axiomatic system which codifies the (feasible) physical theory of constructing geometric figures obtained by certain physical instruments.
  - Points only
  - The construction implied by Eulcid's fifth Postulate is not included.
  - All foundational theorems about such constructions have analogs (does not suffer from poverty).
  - Contains a robust theory of parallel line segments.
  - The intended models are finite.
  - Classical logic

Competing foundations for mathematics: how do we choose?

- Hilbert's program failed (opinion).
- Strict finitism is a philosophical stance desiring indisputable 'concrete' foundations for mathematics.

Having said that,

- Strict finitism is not an agreed upon philosophical stance.
- Very little mathematics had been shown to have strict finite foundations.

A Takeaway?

- Strict finitism can be viewed as something to aspire to.
- Mathematicians can still study 'new' strict finite mathematics.

Further topics for discussion:

- On issues of Vagueness: Sorities Paradoxes: Dummett and Wright
- More details about a finite, axiomatic, complete, inconsistent model of arithmetic
- More details about operationalistic geometry

A Philosophical Issue: Vagueness

- Where is the limit/end of the natural numbers.
- Terminology (or Predicates) like ‘small’, ‘finite number’, or ‘natural number’ are vague.
- They suffer from Sorities paradoxes.
- See Dummett, *Wang Paradox*, 1975
- See Wright, *Strict Finitism*, 1993
  - Devised a semantic proof theory to codify learning histories.
  - Built on by Yamada, *Wright’s Strict Finitism*, 2017

Rich Finitism: work of Priest and Van Bendegem

- Inconsistent Mathematics: Classical mathematical axioms are asserted within a framework of a non-classical logic which can tolerate the presence of a contradiction without turning every statement into a theorem.
  - Principle of Explosion
  - Paraconsistent Logic
- Peano Arithmetic
  - Constant 0, Successor +1, Addition, Multiplication, Induction
- Standard Model of Arithmetic
  - $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
- Inconsistent Finite Model:
  - $\mathbb{N}^* = \{[0], [1], [2], \dots, [L-1], [L, L+1, L+2, \dots]\}$
    - $S(\overline{L}) = \overline{L}$ and $S(\overline{L}) \neq \overline{L}$
    - $S(\overline{0}) \neq \overline{0}$
    - Finite, Decidable, Axiomatic, Complete, Inconsistent

Van Bendegem, *Strict Finitism as a Viable Alternative in the Foundations of Mathematics*, 1994

Gödels First Incompleteness Theorem:

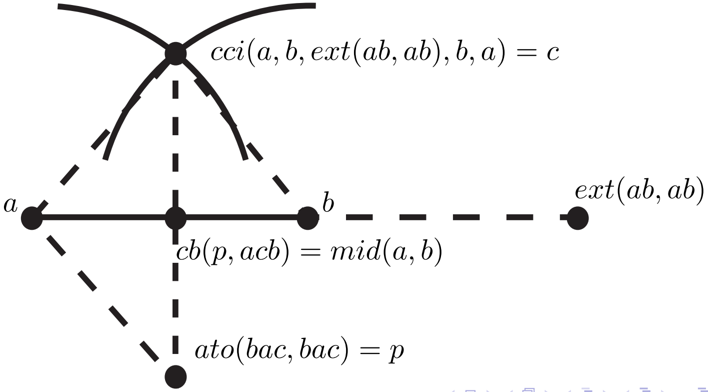
- Any axiomatic theory of arithmetic with appropriate expressive capability is incomplete
- Full version: ... is either incomplete or inconsistent
- Principle of Explosion
- Classically: If the Gödel sentence is false it is also true.
    - Thus the Gödel sentence must be true which implies that it cannot be proved. Thus, arithmetic is incomplete.
- Paraconsistent Logic: The Gödel sentence is true and false without having the Principle of Explosion take hold.

Operationalistic Geometric Constructions:

- Language/Logic: First-Order, Quantifier-free, Points only
- Undefined Relations: Between, Segment Congruence, Angle Congruence, Coplanar (Four Points), Same Angle Orientation,
- Undefined Constructions (One Step): (Directed) Segment Extension, Angle Transport (Same Side), Circle-Circle Intersection, Crossbar, Orthogonal.
- Instruments: Marked Straight Edge, Marked Protractor, Compass, Orthogonal Tool, Flat Disk
- Geometric Configurations: a finite collection of points where all points are either one of two distinct (starting) points α and β or are the result of iterative applications of the five undefined constructions to α and β.
- Diameter: The diameter of a configuration no more than doubles with each application of an undefined construction.

Iterative Constructions:

- (Directed) Segment Extension: $ext(ab, cd)$
- Angle Transport (Same Side): $ats(abc, def)$
- Circle-Circle Intersection: $cci(c_1, a, b, c_2, d)$
- Crossbar: $cb(d, abc)$
- Orthogonal: $o(a, b, c)$



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