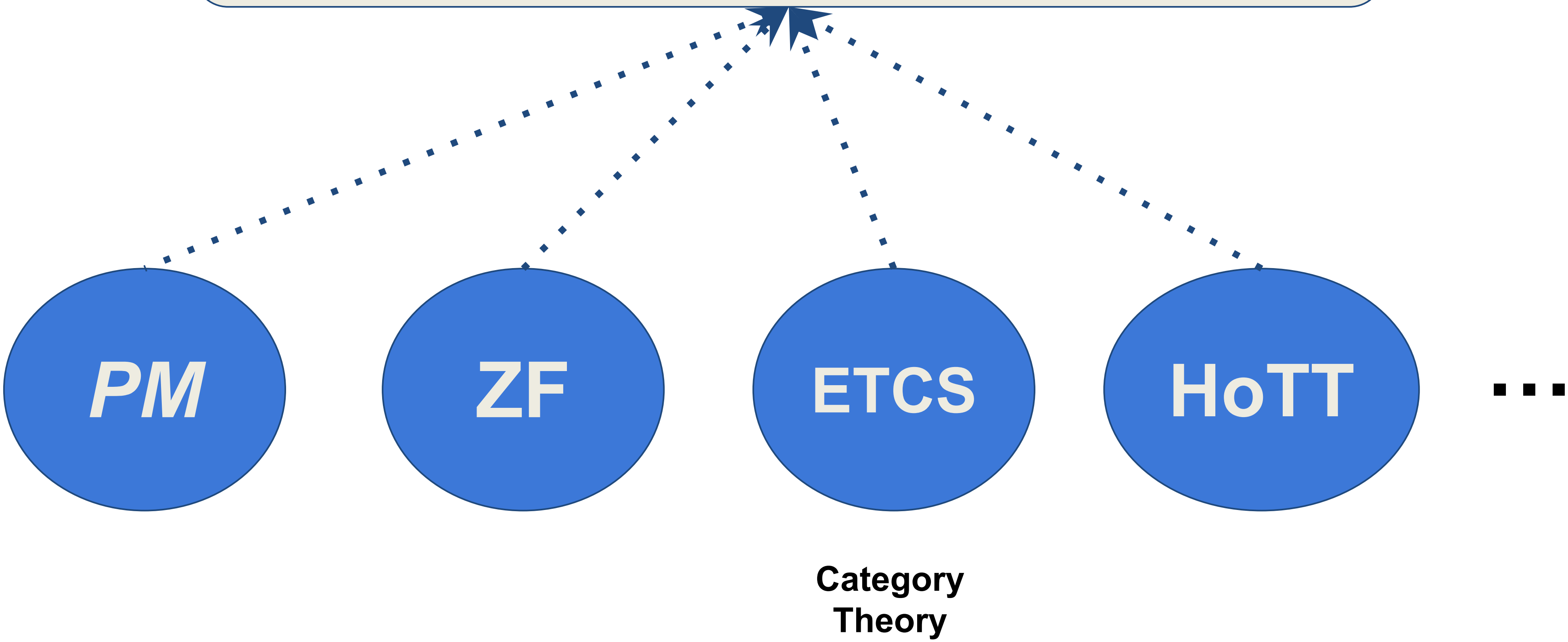
Category Theory as an Explanatory Foundation

Chanwoo Lee
(Department of Philosophy, UC Davis)

AMS Special Session on Mathematics and Philosophy

What’s the “true”, “best” foundation of mathematics?



There is **more than one** sense of ‘foundation’.

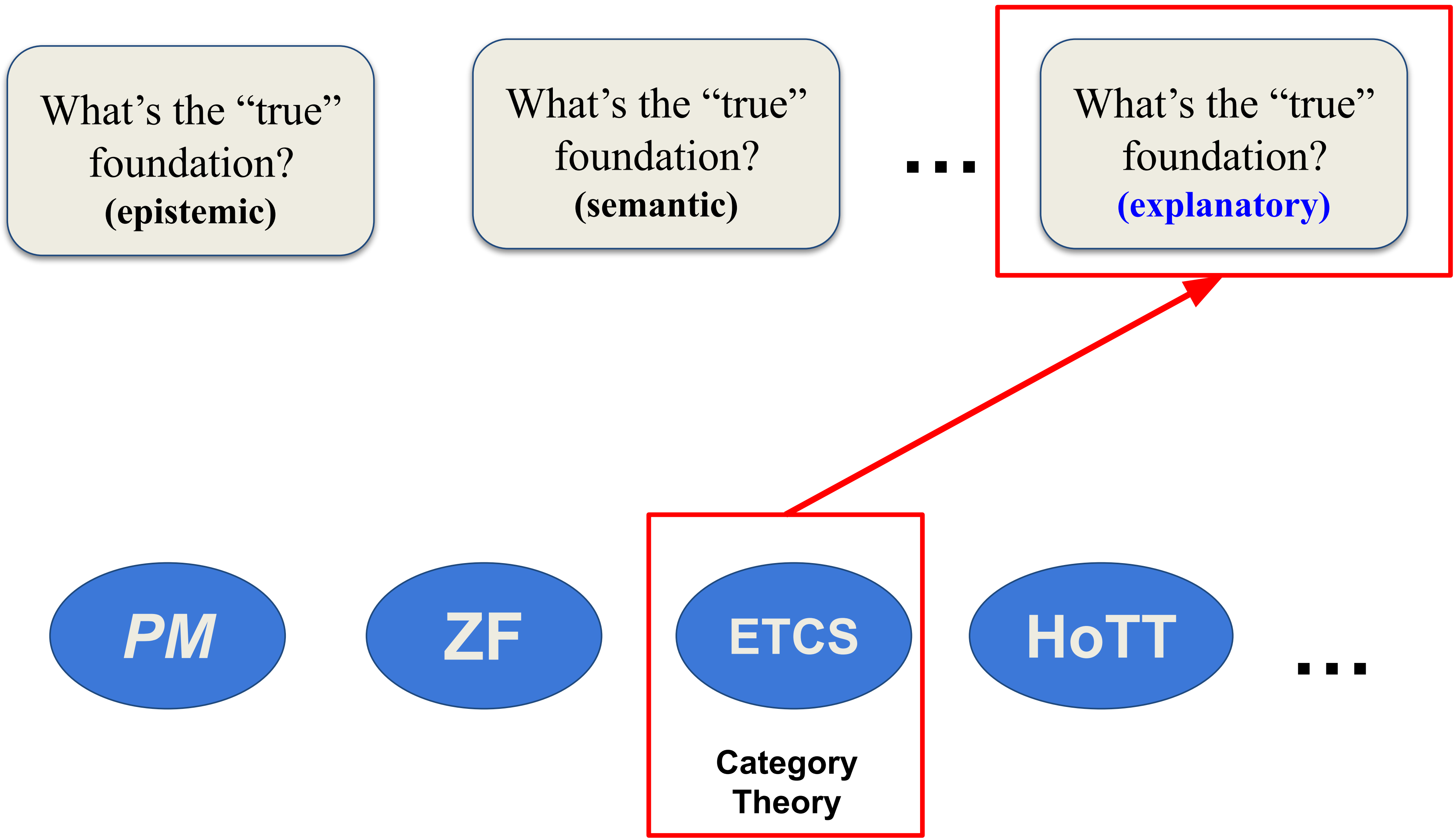
“Psychological”

“Semantic”

“Epistemic”

. . .

“Explanatory”

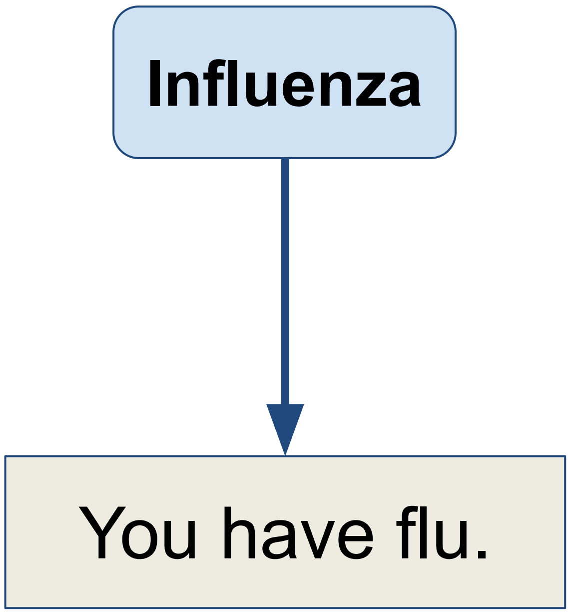

What's the "true" foundation? (epistemic)
What's the "true" foundation? (semantic)
. . .
What's the "true" foundation? (explanatory)
PM
ZF
ETCS
HoTT
. . .
Category Theory

Main claims

- An explanatory sense of foundation is both historically situated and philosophically motivated.
- An appeal of category-theoretic (**CT**) frameworks can be understood in terms of an explanatory sense of foundation.

Explanation?

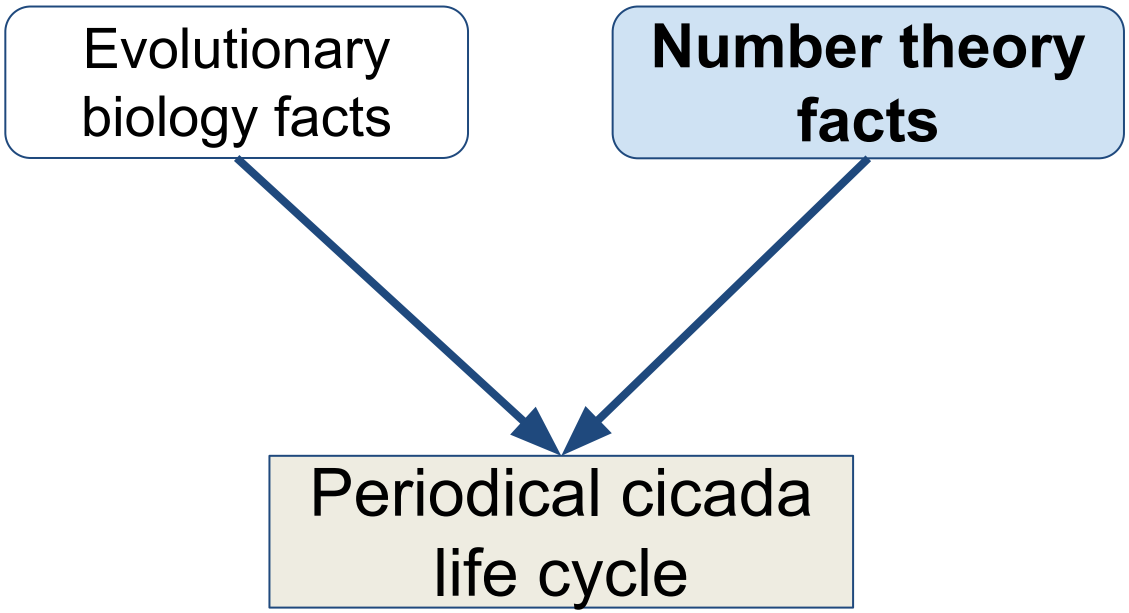
- Q. Why do I have flu?
- A. Because you got influenza virus.


Influenza
You have flu.

Mathematical explanation

Q. Why do periodical cicadas emerge after **13** years or **17** years?

Mathematical explanation



Alan Baker. “Are there Genuine Mathematical Explanations of Physical Phenomena?”

Mathematical explanation

Q. Why are the polynomials of degree five and higher unsolvable in radicals?

The unsolvability of the quintic by radicals

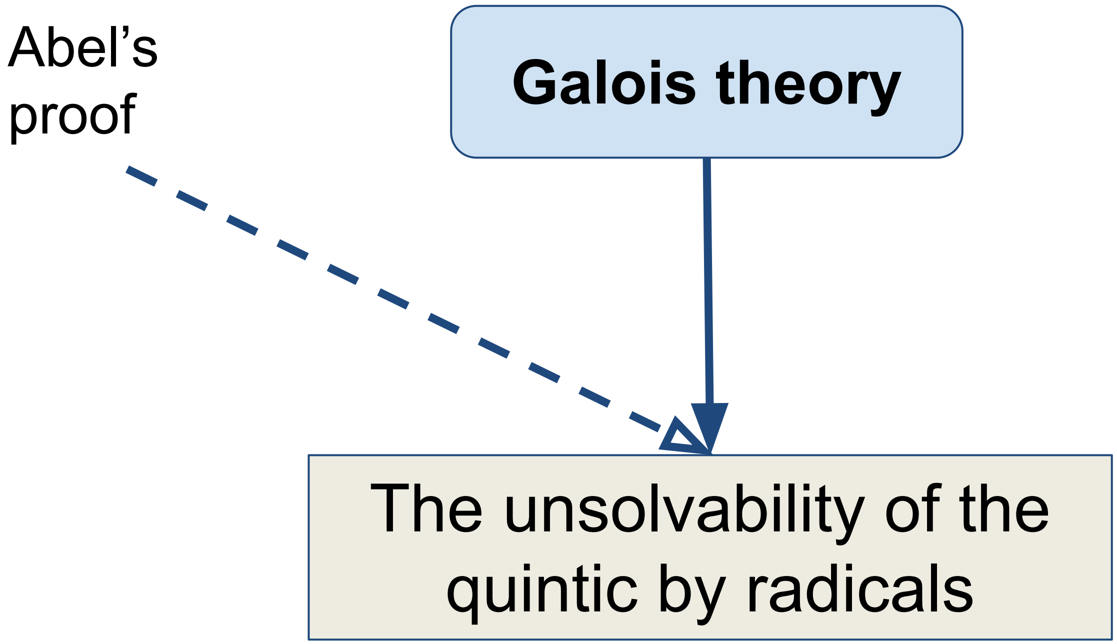
Niels Henrik Abel

(1824)

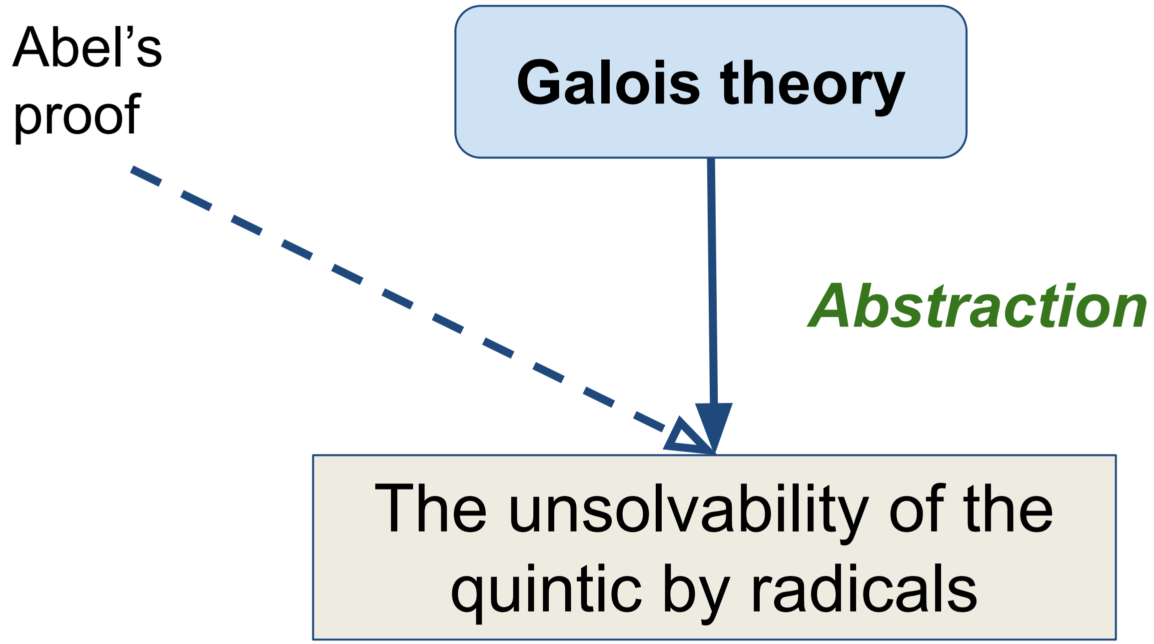
Évariste Galois

(1830/1846)

Mathematical explanation



Mathematical explanation



Christopher Pincock.
"The Unsolvability of The Quintic: A Case Study in Abstract Mathematical Explanation"

Main claims

- An explanatory sense of foundation is both historically situated and philosophically motivated.

- An appeal of category-theoretic (**CT**) frameworks can be understood in terms of an explanatory sense of foundation.

Russell

Gödel

Bourbaki

Lakatos

Russell

Gödel

Bourbaki

Lakatos

Russell

Gödel

Bourbaki

Lakatos

Russell

Gödel

Bourbaki

Lakatos

Russell

Gödel

Bourbaki

Lakatos

Russell

Gödel

Bourbaki

Lakatos

“an inductive method”

“general laws in any other science”

“explanatory [...] as exactly in physics”

“the deep-lying reasons for [...] the common ideas [...] buried under the accumulation of details”

Russell

Gödel

Bourbaki

Lakatos

They all attest to the view that a proper foundation should be explanatory.

Main claims

- An explanatory sense of foundation is both historically situated and philosophically motivated.

- An appeal of category-theoretic (**CT**) frameworks can be understood in terms of an explanatory sense of foundation.

Is CT really explanatory?

The unsolvability of the quintic by radicals

Niels Henrik Abel

(1824)

Évariste Galois

(1830/1846)

19c

20c

Galois Theory

‘The conceptual approach’

The Göttingen school

Category Theory

Riemann

Dedekind

Hilbert

Noether

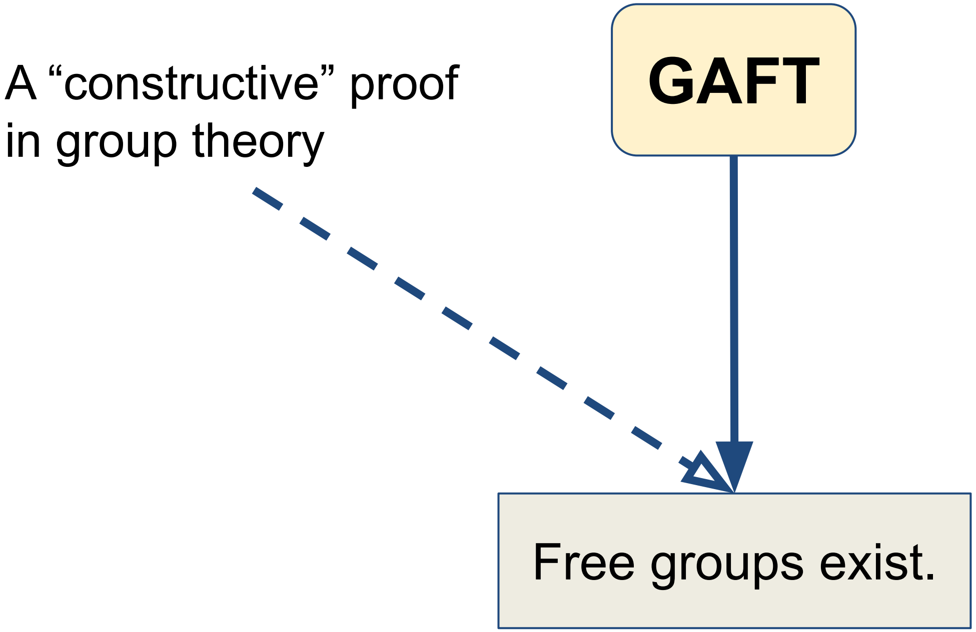
Mac Lane

Case studies on CT explanation

I. Left adjoint functors preserve colimits.

II. An explanation through general adjoint functor theorem (GAFT)

Colyvan, Cusbert, McQueen. “Two Flavours of Mathematical Explanation”



This left adjoint F: **Set**$\rightarrow$**Grp** assigns to each set X the free group FX generated by X, so [GAFT] has produced this free group without entering into the usual (rather fussy) explicit construction of the elements of FX as equivalence classes of words in letters of X.

Saunders Mac Lane. *Categories for the Working Mathematician*

So [GAFT] tells us that, for instance, the free group functor exists. In [earlier examples], we began to see the trickiness of explicitly constructing the free group on a generating set A. […] But using [GAFT], we can avoid these complications entirely.

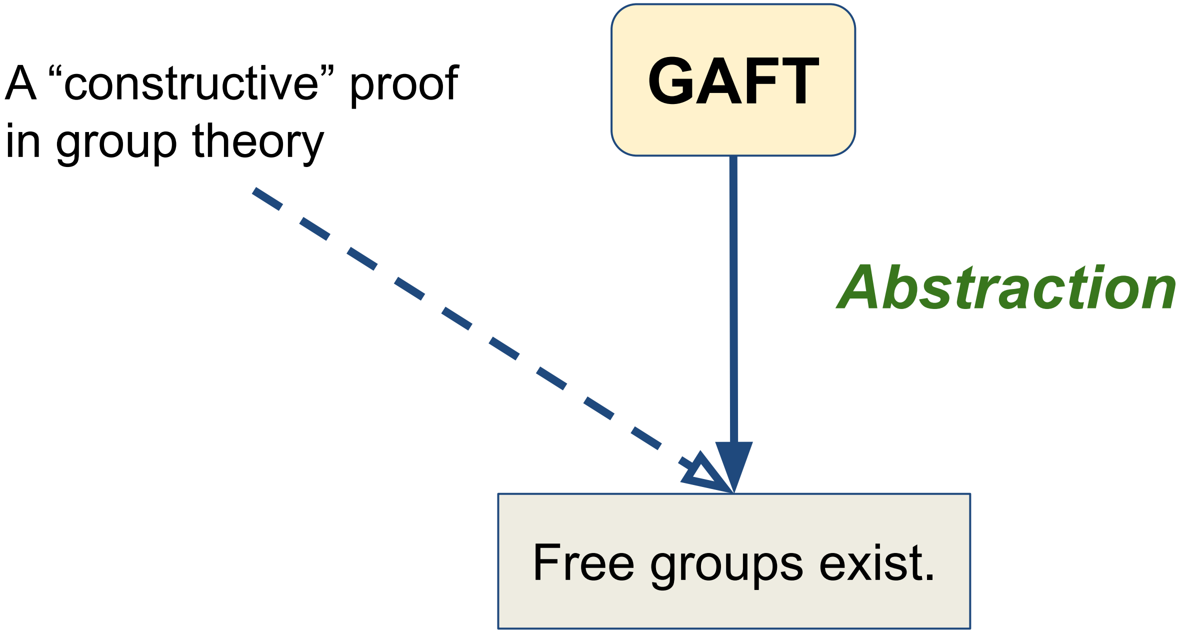
Tom Leinster. *Basic Category Theory*

Colyvan, Cusbert, McQueen. “Two Flavours of Mathematical Explanation”

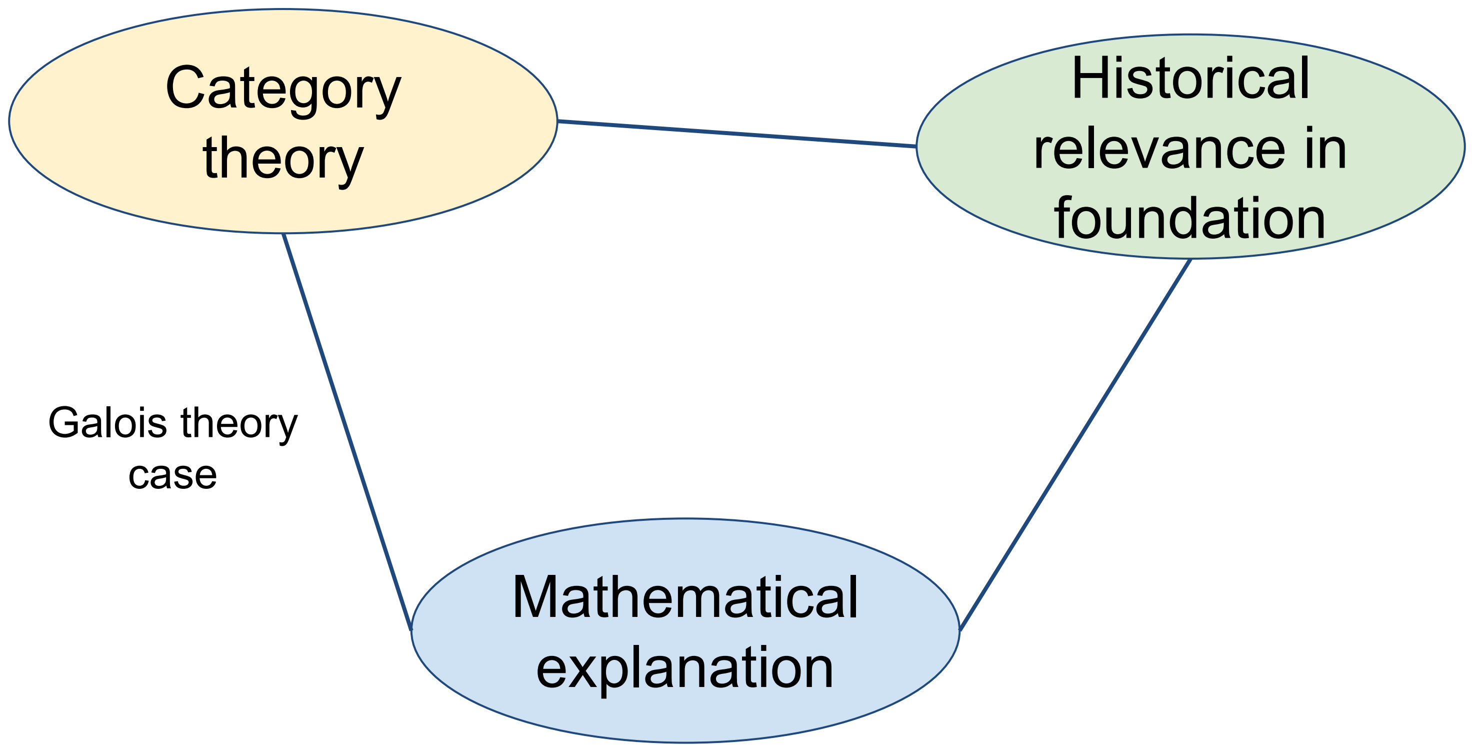
the claimed advantage can also be thought of as an explanatory advantage: [...]

the real reason for the existence of free groups is found at the more abstract structural level.

Colyvan, Cusbert, McQueen. “Two Flavours of Mathematical Explanation”



Wrap-up


Category theory
Historical relevance in foundation
Galois theory case
Mathematical explanation

Russell

Gödel

Bourbaki

Lakatos

in mathematics, except in the earliest parts, the propositions from which a given proposition is deduced generally give the reason why we believe the given proposition. But in dealing with the principles of mathematics, this relation is reversed.

“The Regressive Method of Discovering the Premises of Mathematics.”

Russell

Gödel

Bourbaki

Lakatos

the inferring of premises from consequences is the essence of induction; thus the method in investigating the principles of mathematics is really an inductive method, and is substantially the same as the method of discovering general laws in any other science.

"The Regressive Method of Discovering the Premises of Mathematics."

Russell

Gödel

a decision about [an axiom's] truth is possible also in another way, namely, inductively by studying its "success,"

"What Is Cantor's Continuum Problem?"

Bourbaki

Lakatos

Russell

Gödel

Bourbaki

Lakatos

Professor Gödel suggests that so-called logical or set-theoretical ‘foundations’ for number theory, or any other well established mathematical theory, is explanatory, rather than really foundational, exactly as in physics.

Henryk Mehlberg.
“The Present Situation in the Philosophy of Mathematics.”

Russell

Gödel

Bourbaki

Lakatos

What the axiomatic method sets as its essential aim, is exactly that which logical formalism by itself can not supply, namely the profound intelligibility of mathematics.

"The Architecture of Mathematics."

Russell

Gödel

Bourbaki

Lakatos

Where the superficial observer sees only two, or several, quite distinct theories, lending one another "unexpected support" […] through the intervention of a mathematician of genius, the axiomatic method teaches us to look for the deep-lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details

"The Architecture of Mathematics."

Russell

Gödel

Bourbaki

Lakatos

The battle between rival mathematical theories is most frequently decided also by their relative explanatory power.

"A Renaissance of Empiricism in the Recent Philosophy of Mathematics."

Colyvan, Cusbert, McQueen. “Two Flavours of Mathematical Explanation”

the claimed advantage can also be thought of as an explanatory advantage: whereas the constructive proof bogs down in detail, the category theory proof rises above such details to reveal the real reasons for the existence of free groups. According to this line of thought, the real reason for the existence of free groups is found at the more abstract structural level.

Wrap-up

- There *is* mathematical explanation, e.g., abstract explanation via Galois-theoretic proof.
- Historically, explanation has been foundationally relevant.
- The same abstract explanatory account can be applied *mutatis mutandis* to category-theoretic results.
- Thus, category theory is foundational in an explanatory sense.