An unorthodox Philosophy of Mathematics

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Competing Foundations for Mathematics. How do we choose?

- it is possible to compare all the different positions in the Philosophy of Mathematics according to a well defined set of criteria <u>agreed among all</u> the scholars?

- there are many different possible criteria that are pertinent
  - it would be almost impossible to agree on some of them
    - many criteria can not be compared with each other
      - because they are intended for completely different purposes

The experience of teaching and learning mathematics

- I’m very interested in the teaching and learning of mathematics,
- for me, a valid criterion on how to choose among different positions is their adequacy to explain and represent my experience in the teaching and learning of mathematics,
- I wouldn’t pretend this to be a universal criteria, valid also for other people.

Consequences of this particular point of view

- The several positions can be analyzed from this point of view.
- To do this briefly I'll risk being blunt by oversimplifying most tendencies in the Philosophy of Mathematics to a single statement about mathematics.
- Of course, I'm not pretending that a single statement can summarize a whole position about mathematics,
- I hope that these statements will work as a quick reminder of key aspects of each one, and these aspects will then be analyzed <u>from the point of view of a teacher trying to ground his/her experience in the teaching and learning of mathematics.</u>

Mathematics is dependent on each individual's accumulated experience (Empiricism.)

- Mathematical entities are not directly experienceable through our senses.
- Nobody has ever run across Mr. Number on the street to have a quick chat.
- The things that we actually experience seem to be very distant from the un-concreteness and complexity of mathematical notions.

Mathematics is just for some geniuses (Inborn notions.)

- The idea that many important and fundamental notions of mathematics should be inborn, leave us teachers with a very difficult and uncomfortable question:
  - What should we do with students that weren't lucky enough to be born with these notions?
  - Shouldn't we be trying to test little children to early identify those without these inborn notions in order to avoid torturing them with years of hopeless teaching of mathematics.

Mathematics is just for those who can access the “abstract” world of the ideas (Platonism.)

- Trying to guide students in the “discovery” of a pre-constructed and immutable world of ideas results in bad teaching,
- teachers should rather help each student in the personal construction of every notion.
- Constructivism is a strong and well grounded teaching theory that over the years has accumulated an overwhelming amount of evidence.
- Constructivist teachers have a very difficult task if they try to reconcile this teaching attitude with platonism in mathematics.

Mathematics talks about “abstract” entities and it is convenient to consider them as objects of which it’s not relevant to find out their nature (Practical Platonism.)

- It's strange to think that I could teach something the nature of which is not important to me.
- On the contrary, deepening on the nature of the taught object should allow me to better understand its peculiarities and the possible difficulties that could arise during the acquisition of such a notion.
- On the other hand, students tend to be very curious about the nature of mathematical objects.

Mathematics is pure, absolute, independent from concrete experiences (Idealism.)

- Rather than pure and absolute notions, during the learning process it is not uncommon to deal with provisory notions that may be messy and partial.
- Denying any connection to students' concrete experiences makes it hard, if not impossible, to convey any significance to mathematics.

Mathematics is based on axioms that are transparent and evident, from which everything else can be deduced (Logicism.)

- Transparency and self-evidence are immediately challenged in the learning-teaching environment.
- Experience suggests that it is not until a whole theory is mastered (by some non-deductive path) that the primitive notions' significance is truly grasped and that the justifications of the axioms are understood.

Mathematics is purely a game of symbols (Formalism).

- The idea that points and lines may be replaced by beer jars and tables, provided they respect the due relationships among them, is correct and useful in some settings.
- But pretending that mathematical objects are meaningless is very problematic in order to understand why mathematics should be learned.
- Why don’t we teach chess instead?

Mathematics is a convention that should be known (Conventionalism.)

- Why should students learn de capricious agreements of the mathematical community?

Mathematics is a fictional story (Fictionalism.)

- The same observations made in the case of platonism would do in this case too,
- Our remarks are about who can access the mathematical ideas.

A very difficult standard to be met

- The compatibility of most of the competing foundational positions of mathematics with the teaching and learning experience is very problematic:
    - some of them rely on key assumptions that are contradictory to well accepted theories about teaching and learning,
    - others fail to justify the very need for teaching mathematics to the population: not every student will become a mathematician.
- Maybe <u>something different is needed</u> to adequately fulfill these expectations,
    - even if it is not something completely new, as <u>some kind of correction</u> for some of the previous positions could suffice.

Empiricism pros

- From the teaching and learning perspective the emprisit idea that the significance and certainty of non-concrete notions are founded in the palpable certainty of basic perceptions is an appealing assertion for us:
    - it provides a clear connection to the real world, <u>justifying the significance and applicability</u> of the mathematical notions that have been transmitted for generations;
    - it would mean that, in principle, <u>everyone is capable of learning</u> them;
    - it <u>doesn’t need to make recourse of dubious assumptions</u> difficult to be made compatible with teaching and learning theories.

Empiricism cons

- There is a huge leap in its explanation about how non concrete notions, like mathematical objects, can be constructed starting from basic perceptions.
- The lack of a detailed understanding of the passage from basic and certain perceptions to at least a set of the most elementary notions that can be built upon them, is a crucial setback from the teaching and learning point of view,
    - it means skipping our main interest: the process, rather that the result.

Our attempt at fulfilling this gap

- We have undertaken a new line of work in the foundations of mathematics, aimed at fulfilling this gap.

- It seems feasible for us to show in detail a possible path from basic perceptions to some of the elementary mathematical notions.

internal senses

- The perceptions experimented by humans are not only from the five classical senses, but also from the internal senses, for example:
  - the perception of being oneself,
  - the perception of intentionally performing an action,
  - the perception of one’s own posture,
  - etc.

The similarity of the perceptions felt by different individuals

- The problem with internal senses is that the perceptions produced by them are completely <u>certain but known only by the individual</u> perceiving them.
    - The same is true also for the five classical external senses, but, in this case, it is possible to identify perceptions received by different individuals according to the environment in which they are generated:
        - in the same environment, it is assumed that the perceptions perceived by different individuals are the same, even though we have no way on knowing that, for no one can enter the interiority of another individual.
- This problem leads to one fundamental assumption: <u>the similarity of the perceptions felt by different individuals</u>, assumption that is captured by the attitude of putting oneself in other people's shoes.

Initial building blocks

- Basic perceptions
  - Coming from external senses
  - Coming from internal senses
- Memory: as the possibility of recording perceptions in a structured manner and accessing them afterwards.
- Mental operations: as the possibility to make transformations to what is being perceived whether directly or indirectly through the memory. The result of a mental operation is a new perception that can be used by a further operation or stored. We try to keep the number of different operations as small as possible.

Mental operations

- Abstraction: considering only certain aspects deemed relevant while disregarding the rest.
- Comparison: compare sets of perceptions to see which aspects are the same.
- Generalization: when an aspect is found in every set of perceptions of a certain type that has been experienced, by this operation is assumed present in all sets of perceptions of that type, even if they weren't experienced.
- Idealization by transfer: add to a set of perceptions a new aspect that has never been perceived alongside the others, but in a completely different setting, maybe because it is considered a desirable aspect.
- Idealization by negation: add to a set of perceptions the negation of one aspect, maybe because it's considered undesirable or problematic, even if it's not clear which are the consequences of the negation of an aspect.

Manageable models of reality

- The complication of the totality of the perceptions perceived by the individual forces to accept that it is impossible to know the reality totally and exactly,
- instead humans refer to a model of reality that each one builds within himself.
- In a model of reality some organization of the data is introduced making the model more and more complex as it develops to approximate reality better and better:
    - models translate complication into complexity introducing organization among data.

Manageable models of reality

- Using interpersonal communications, the models of reality become <u>intersubjective</u> and are made
  - more and more <u>faithful and comprehensive</u> of what is perceived,
  - still <u>maintaining their manageability</u>,
    - which makes it acceptable, but
    - clearly departing from what is perceived.

Manageable models of reality

- Humans construct models of reality to conveniently act and foresee the effect of their action.
- The effect of the action of one individual is negligible, but the cooperation among humans, even over the times, becomes effective and efficient.
- To obtain cooperation, one should justify the planned actions: thus, a good model of reality should allow us to understand, justify, and forecast the dynamic of reality.

Managing multiplicity

- The quantity of different data and the need to simplify situations leads to deal with their multiplicity
- organizing it independently of who are the elements involved and of any other information about them.
- This human endeavor is what we call ...

...mathematics

Bibliography

The possibility of developing mathematics within this improved empiricist philosophical framework has been outlined in some papers.

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