Peter Epstein
January 5, 2023

A PRIORI CONCEPTS IN EUCLIDEAN PROOF

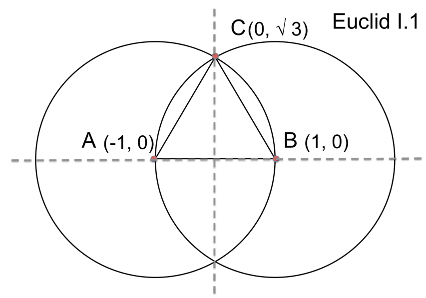
How should we understand the practice of Euclidean proof? I will argue that:

(a) Euclidean proof is not a purely formal system of deductive logic, but one in which the content of specifically geometrical concepts plays a central role.
(b) Our grasp of this geometrical content is *not* derived from experience.

1. Euclidean Proof as a Formal Deduction System

In order for a proof to be valid in a formal deduction system, each step must be derivable from the preceding steps via inferences that take account only of the *form* of the propositions in the earlier steps, without relying on the *content* of the non-logical terms.

- Famously, many of Euclid's proofs fail to satisfy this constraint:



The Gap: What licenses the assumption that the point of intersection, *C*, exists?
The Intended Model (*continuous real plane*): $A(-1, 0)$; $B(1, 0)$; $C(0, \sqrt{3})$
The Defective Model (*merely-dense rational plane*): $A(-1, 0)$; $B(1, 0)$; *C* does not exist

We need a way to rule out the Defective Model:

- The formal features of the axioms, as stated by Euclid, fail to do so.
- The additional logical structure needed is only developed in the 19th c.

Conclusion: Euclid's inferences rely on more than just the *form* of the axioms; the *contents* of our geometrical concepts play a crucial role.

- In particular, our concept of a circle as a genuinely continuous curve rules out the merely-dense Defective Model of *Elements* I.1.
- What is the source of this non-formal content? Is it derived from experience?

2. Strawson's "Picturable Meanings"

Strawson: Euclid's axioms concern our "phenomenal geometry," – the geometry of our visual experience and imagination. Within this phenomenal geometry, "picturable meanings" supply the content needed to plug the gaps in Euclid's proofs.

- In I.1, the "picture of the sense of the description rules out" any models in which the circles fail to intersect.

But there *is* a picture of the construction in which the circles fail to intersect: the picture of the Defective Model, which is *visually* identical to the picture of the Intended Model.

A possible empiricist response: Can the needed distinction be derived from experience via *dynamic imagination*?

- Can distinguish Descartes' chiliagon and myriagon in this way.
- But can't distinguish the Defective Model from the Intended Model.

Conclusion: No "picturable meaning" can fill the gap in Euclid's proof.

3. Visual Limits

Wright: "The kind of idealization involved in the notion of perfect circularity… corresponds to a movement to the limit of a scale, as it were, whose intermediate values are ordered by a comparative – 'is more circular than'."

- But what *is* this limit, if "circular" is a *perceptual* concept?

Proposal: To determine the limit to which a *perceptual* concept approaches:

1) Abstract from contingent imperfections of visual capacities and circumstances.
2) Ask: Given such an idealization of visual experience, what kinds of properties could make a difference to visual experience?

We can thus distinguish, *via* imagination, a 1 nm-gapped curve from a continuous one.

- But can we distinguish continuous curves from merely dense ones?

Perceiving or imagining a gap involves perceiving a point, *p*, and the *next* point, *p'*, along a curve, together with the empty space separating them.

- The gaps in a merely dense curve are not like this; they do not fall between any two points in particular.
- Thus, the gaps in merely dense curves are not something we can grasp via experience or imagination, however idealized.

The distinction between dense and non-dense curves thus marks off a kind of visual limit, allowing us to derive from experience a separation between:

- Curves we can visually experience or imagine as gappy, because they have gaps falling between two specific points (i.e., *non-dense* curves)
- Curves we can't visually experience or imagine as gappy, either because they are genuinely continuous, and so lack any gaps; or because they are merely dense, and so have gaps that don't fall between any two specific points (i.e., the full range of *dense* curves)

But the distinction needed to ground *Elements* I.1 is not this distinction – it is a distinction *within* the category of dense curves, between genuine continuity and mere denseness.

Conclusion: The content that plugs the gap in Euclid's proof—our concept of a circle as a genuinely continuous curve—cannot be derived from (even fully idealized) experience.

4. Conclusion

1) Euclidean proof utilizes a set of contentful, specifically geometrical concepts.

2) The content in question is *a priori* - our concept of a circle as a continuous curve can't be derived from experience.