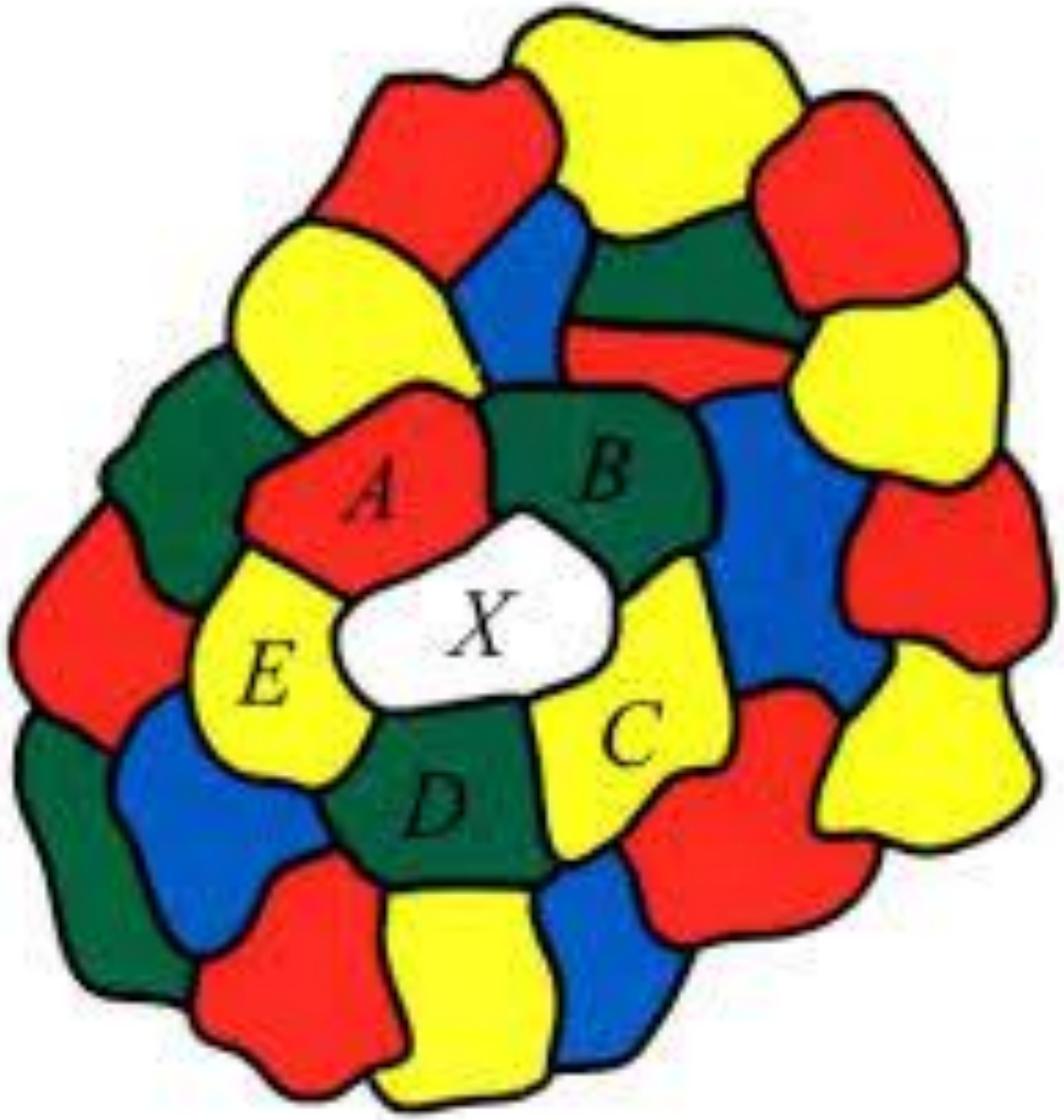
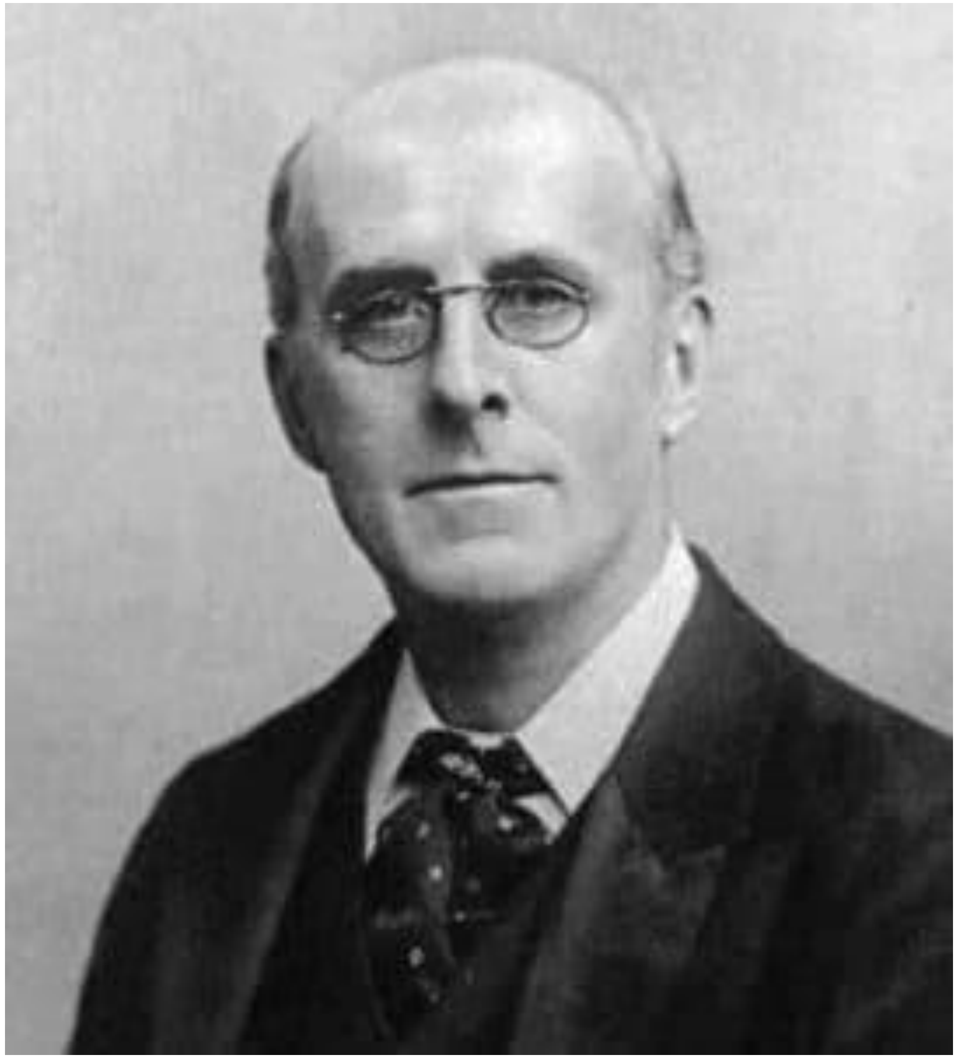
Is it Safe for Mathematicians to Consume Fugu?

Don Fallis

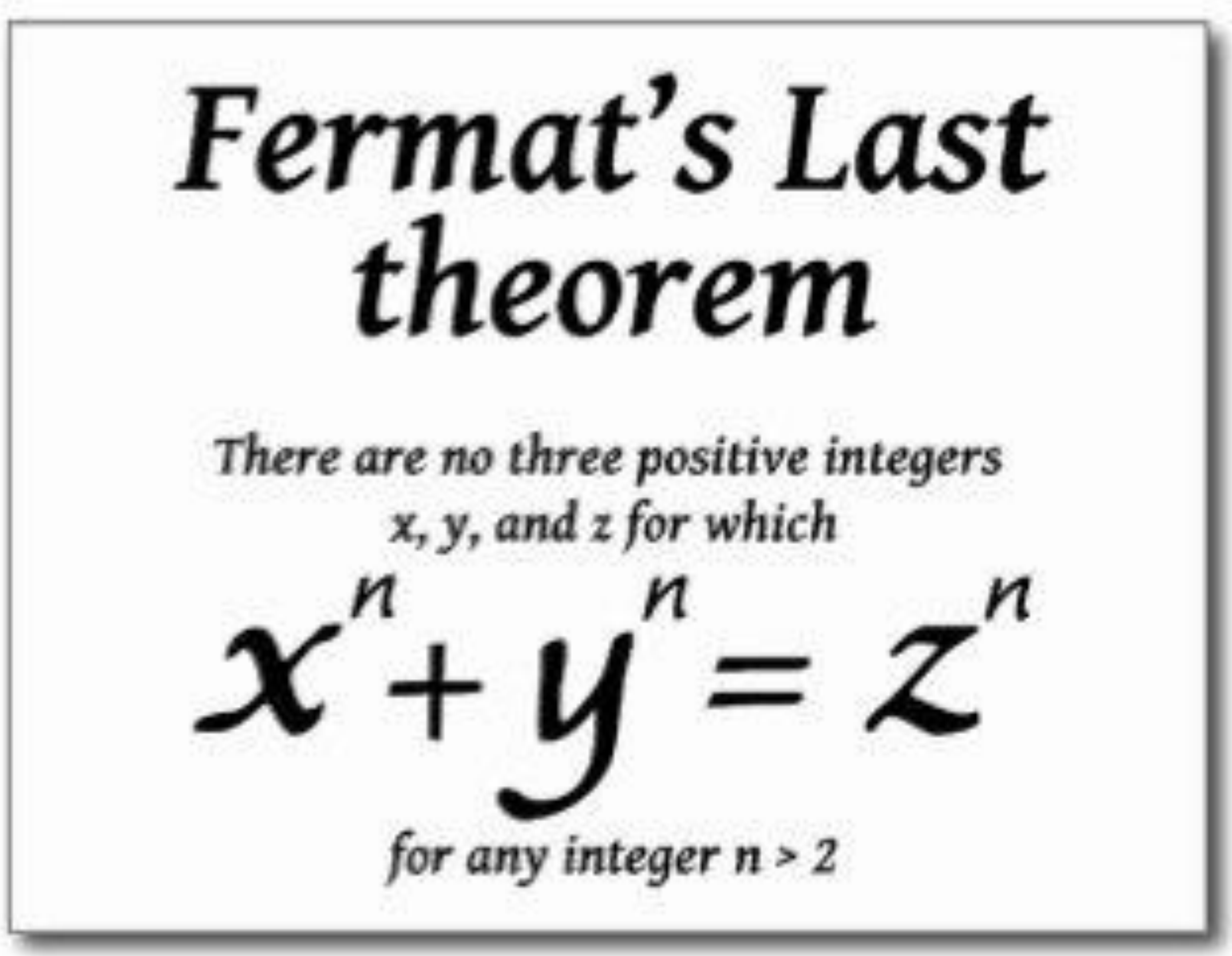
Northeastern University

Kempe's 1879 "Proof" of the Four-Color Theorem





Wiles’s 1993 “Proof” of Fermat’s Last Theorem

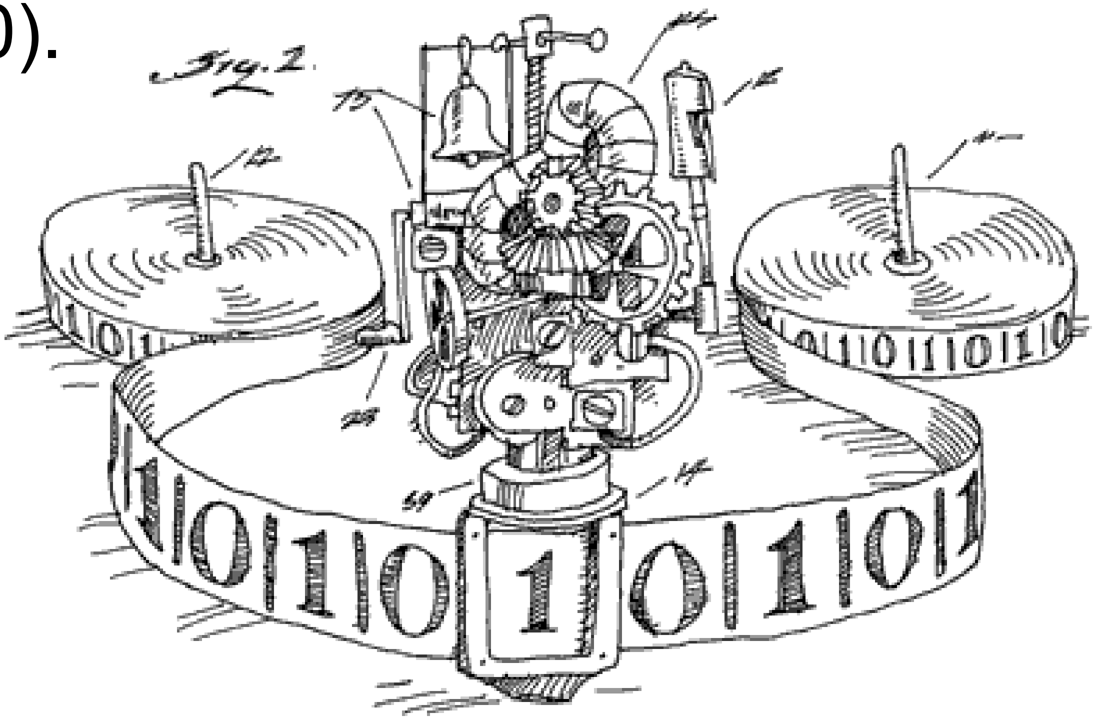


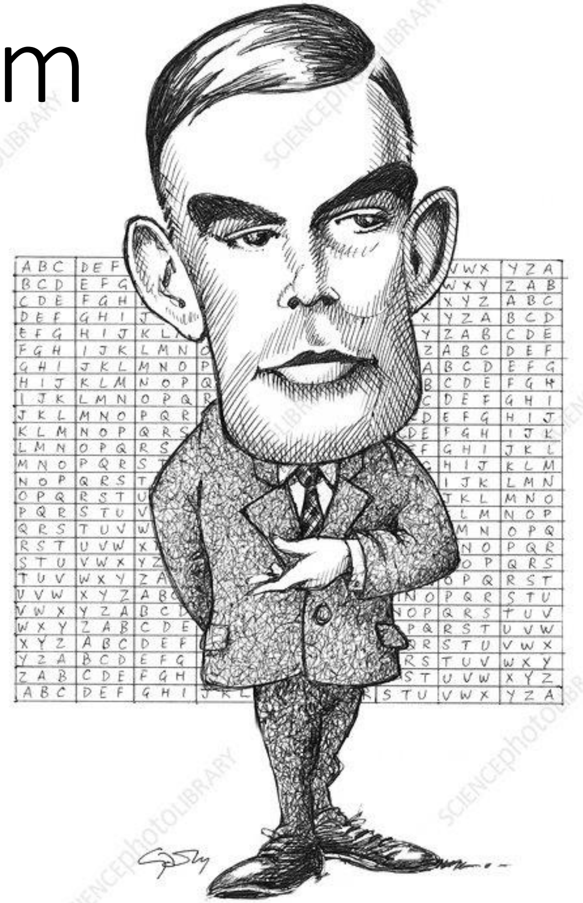
Perelman 2003 on the Poincaré Conjecture

- “Regarding the proofs, [Perelman’s 2002 and 2003 papers] contain some incorrect statements and incomplete arguments, which we have attempted to point out to the reader. (Some of the mistakes in [the 2002 paper] were corrected in [the 2003 paper].) We did not find any serious problems meaning problems that cannot be corrected using the methods introduced by Perelman” (Kleiner and Lott 2008).
- Moreover, “full credit [for] proving Poincaré’s conjecture goes to Hamilton and Perelman” (Cao and Zhu 2006).

Turing 1937 on the Entscheidungsproblem

- “Many of the technical details are incorrect as given” (Davis 1965).
- “Although I do a calculation, I do it in a hurried, slipshod fashion, taking risks … These admissions lay me open to lectures on the subject of my vicious ways” (Turing 1950).





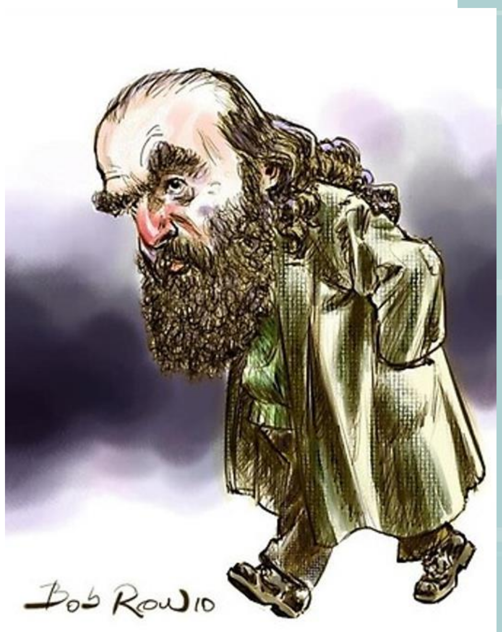
Sorensen's Definition of Fugu

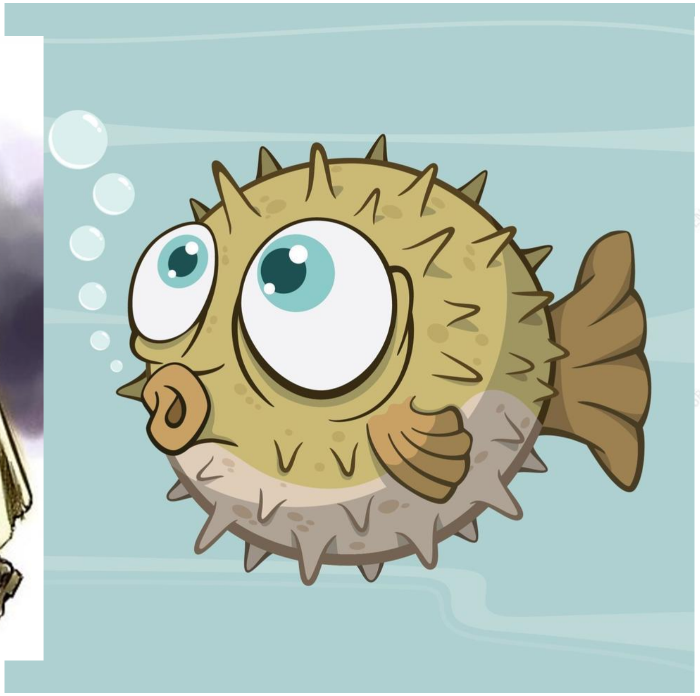
- "A ***lemma*** is a valid intermediate inference."
- "A ***paralemma*** purports to be such a stepping-stone but is actually fallacious."
- "A valid argument that invites a paralemma will be called a ***fugu***."

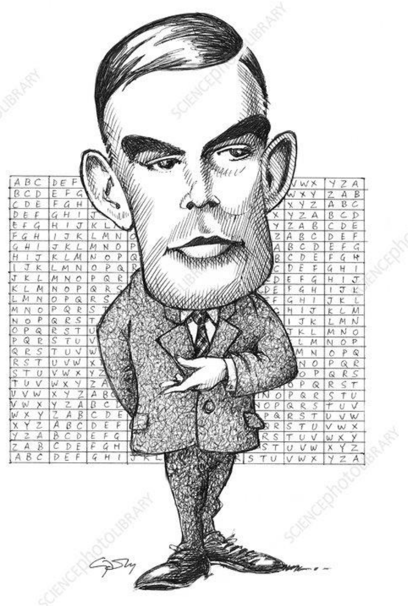
Descartes on Deductive Proof

- "Often people who attempt to deduce a conclusion too quickly and from remote principles do not trace the whole chain of intermediate conclusions with sufficient accuracy to prevent them from passing over many steps without due consideration. But ***it is certain that wherever the smallest link is left out the chain is broken and the whole of the certainty of the conclusion falls to the ground.***"

Fugu Sometimes Yield Knowledge





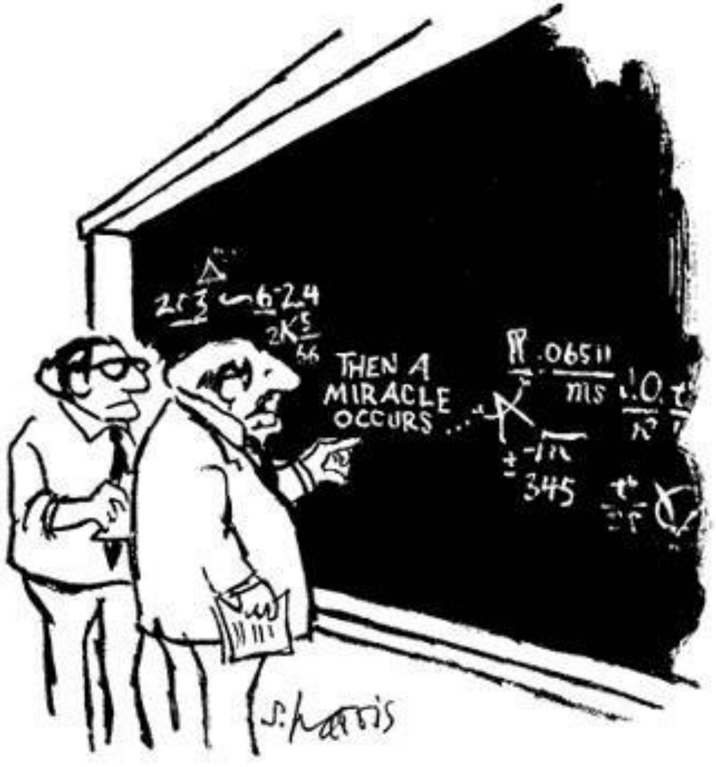


- “Normally, a fugu will not yield knowledge from known premises. But if the reasoning is only slightly fallacious, they do yield knowledge.”
- “Those who eat fugu soup are stupid. But those who don’t eat fugu soup are also stupid.” – Japanese proverb

Ways to Acquire Knowledge of Mathematical Facts

Probabilistic Methods

Testimony



"I think you should be more explicit here in step two."

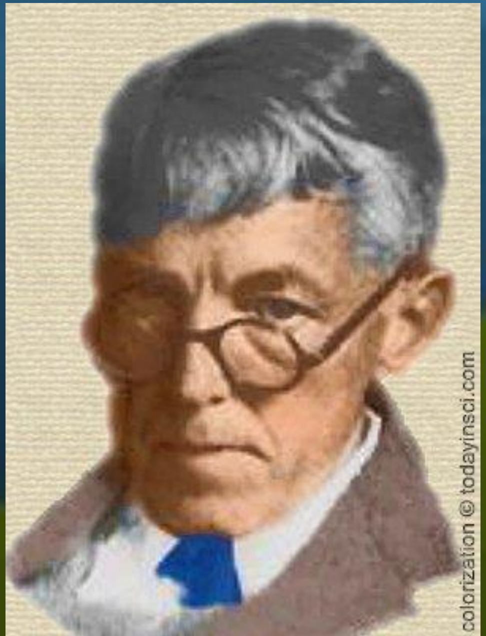
Deductive Proof

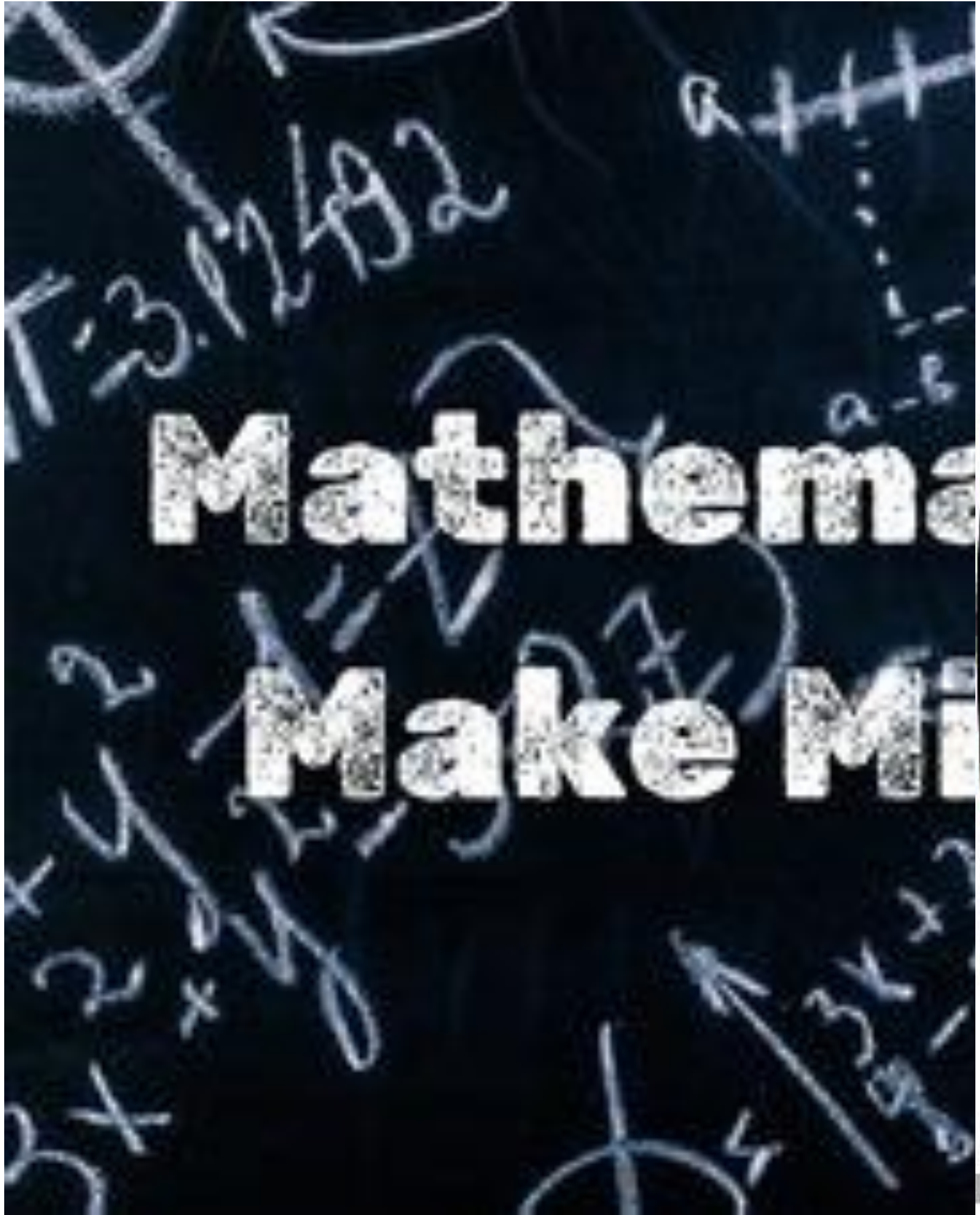
Some Examples of Deductive Knowledge

- A lily patch in Lake Shoji doubles in size each day.
- It takes 48 days for the patch to cover the entire lake.
  - When will ***exactly half*** of its surface be covered?
- Lake Shoji will have ***exactly half*** of its surface covered in ***47*** days.
  - Will ***less than half*** of its surface be covered in ***23*** days?
- Lake Shoji will have ***less than half*** of its surface covered in ***23*** days.

Proving a Theorem is like Following a Path

- “I have myself always thought of a mathematician as in the first instance an *observer*, a [person] who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A, and following it to its end he discovers that it culminates in B. B is now fixed in his vision, and from this point he can proceed to further discoveries” (Hardy 1929).

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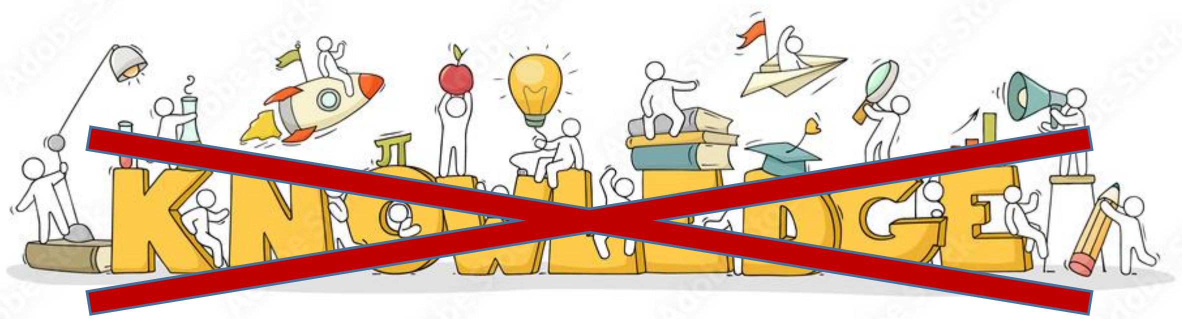
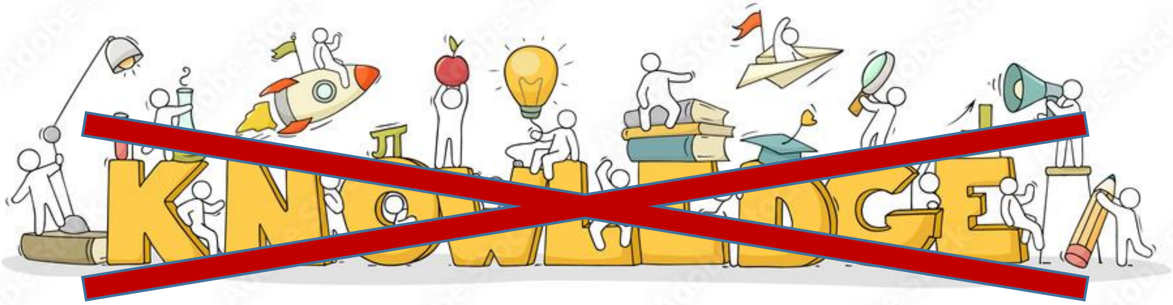
The Invalid Lily Pad Case



- A lily patch in Lake Shoji doubles in size each day.
- It takes 48 days for the patch to cover the entire lake.
  - When will ***exactly half*** of its surface be covered?
- Lake Shoji will have ***exactly half*** of its surface covered in ***24*** days.
  - Will ***less than half*** of its surface be covered in ***23*** days?
- Lake Shoji will have ***less than half*** of its surface covered in ***23*** days.

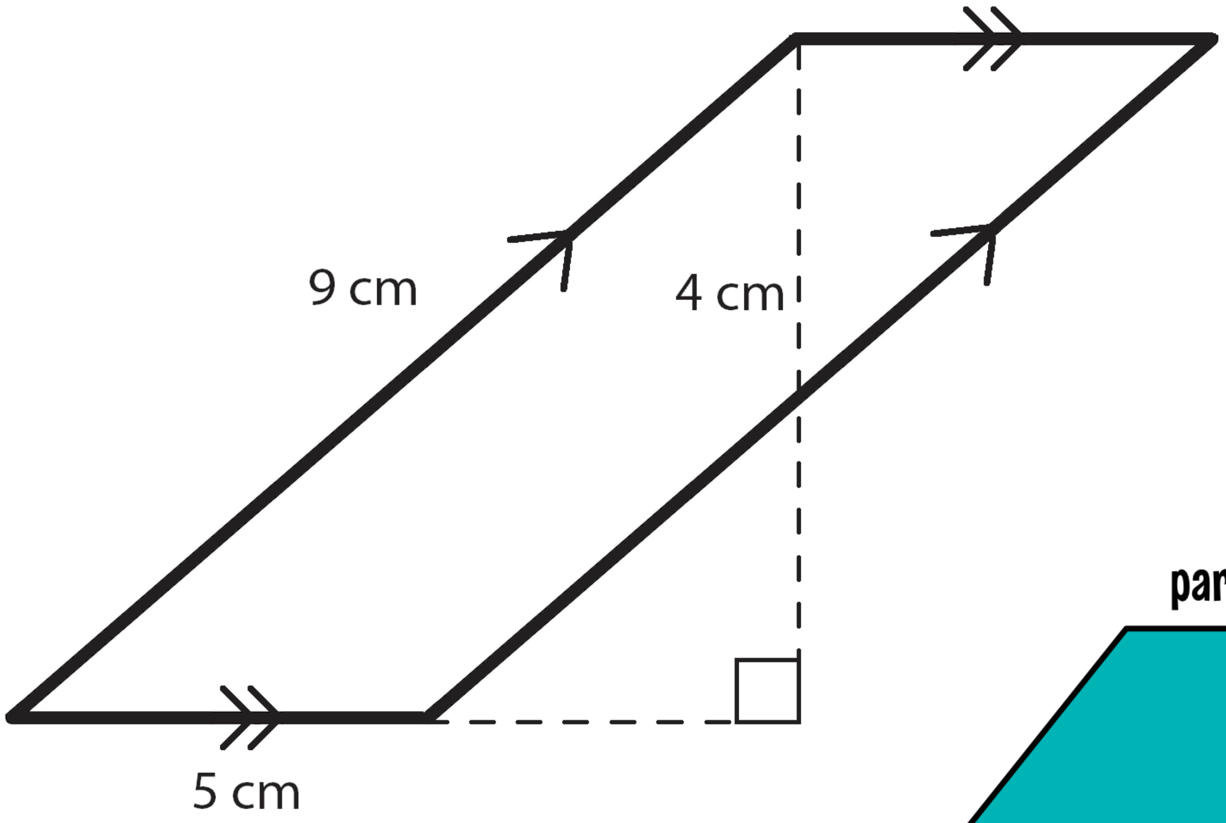
A Slight Variation on the Invalid Lily Pad Case

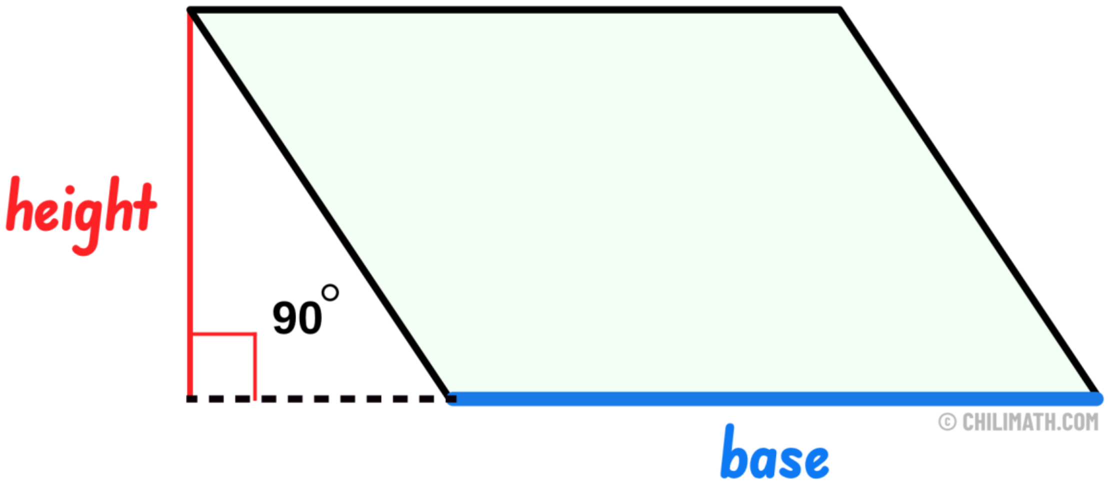


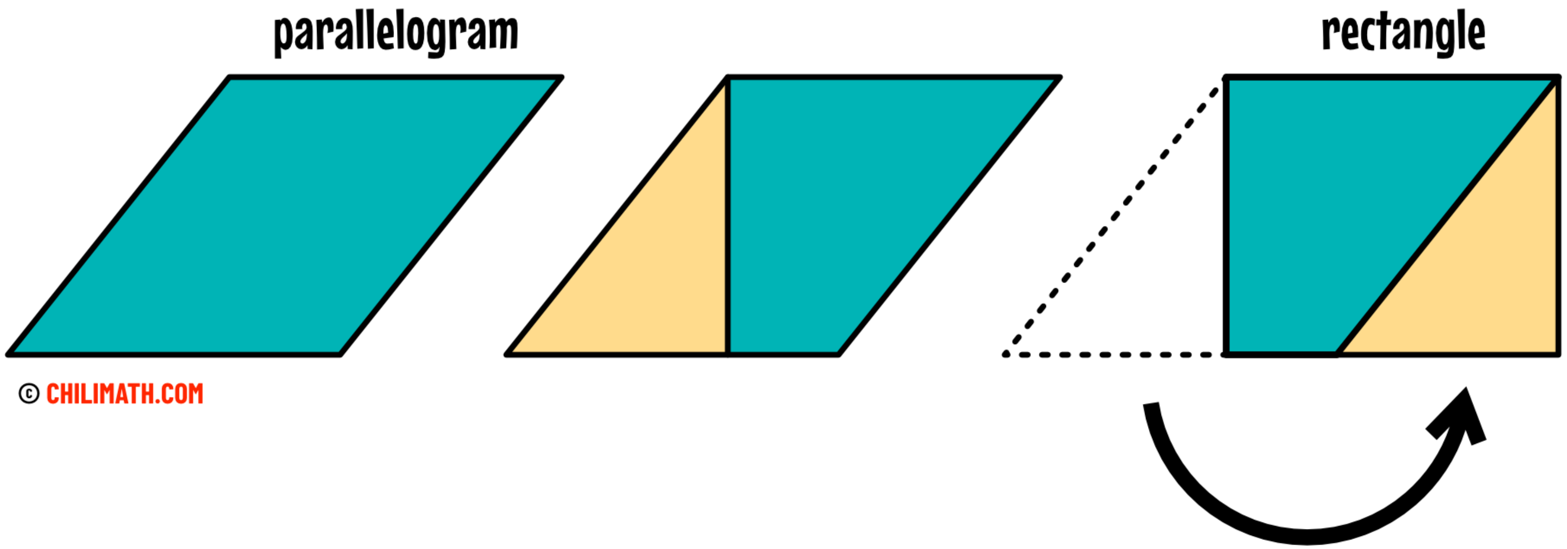
- A lily patch in Lake Shoji doubles in size each day.
- It takes 48 days for the patch to cover the entire lake.
  - When will ***exactly half*** of its surface be covered?
- Lake Shoji will have ***exactly half*** of its surface covered in ***24*** days.
  - Will ***more than half*** of its surface be covered in ***36*** days?
- Lake Shoji will have ***more than half*** of its surface covered in ***36*** days.

Area of a Parallelogram

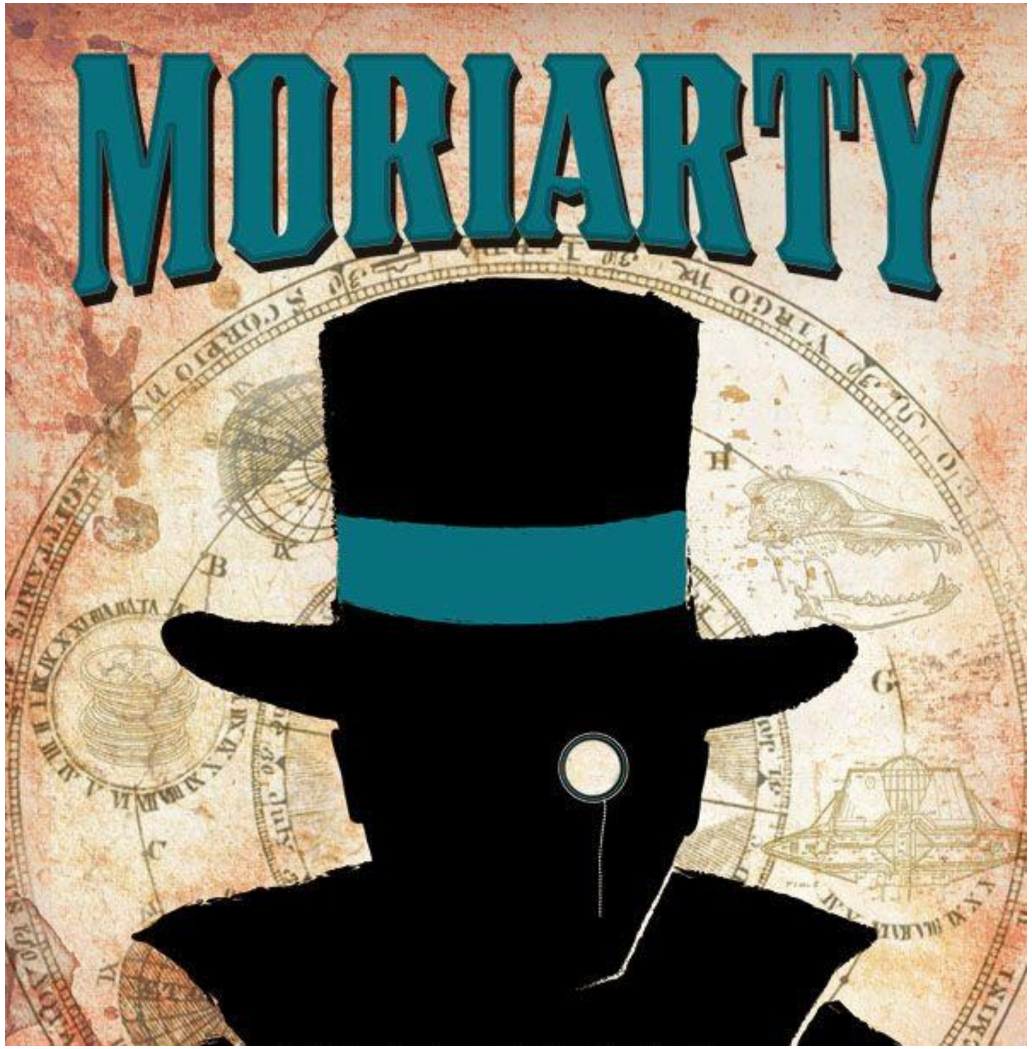
Area(A) = base × height







Propositional Justification versus Doxastic Justification



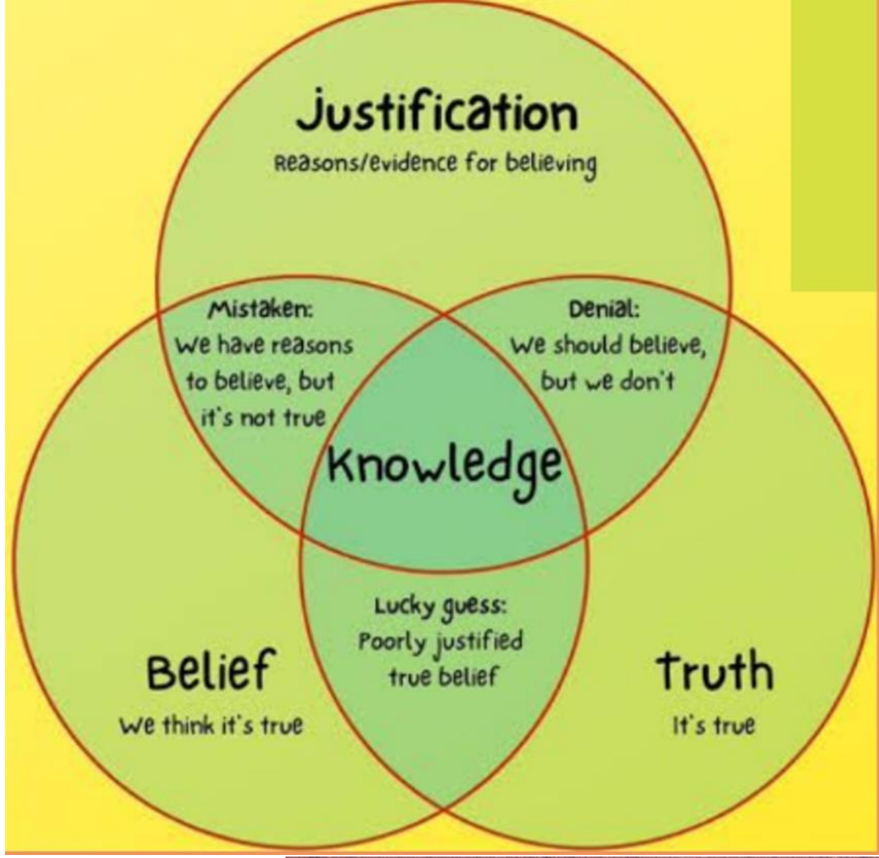
Holmes Deduces **Watson Guesses**

Hume on Fallible Mathematical Justification

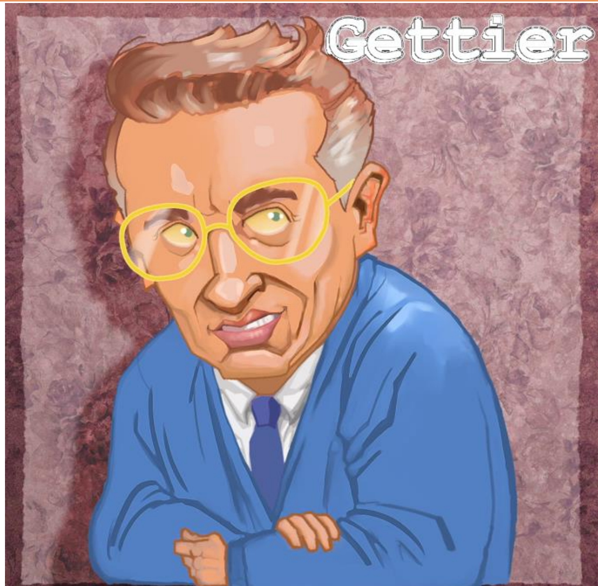
- "There is no Algebraist nor Mathematician so expert in his science, as to place entire confidence in any truth immediately upon his discovery of it, or regard it as any thing, but a mere probability. Every time he runs over his proofs, his confidence encreases; but still more by the approbation of his friends; and is raised to its utmost perfection by the universal assent and applauses of the, learned world."

Traditional Definition of Knowledge

- **S** knows that *p* if and only if …
    - **S** believes that *p*.
    - *p* is true.
    - **S** is justified in believing that *p*.



- But knowledge is not simply justified true belief.
- The justification needs to be ***appropriately connected*** to the truth.
- Otherwise, we are just ***lucky*** that the belief is true.



The Ten Coins Case

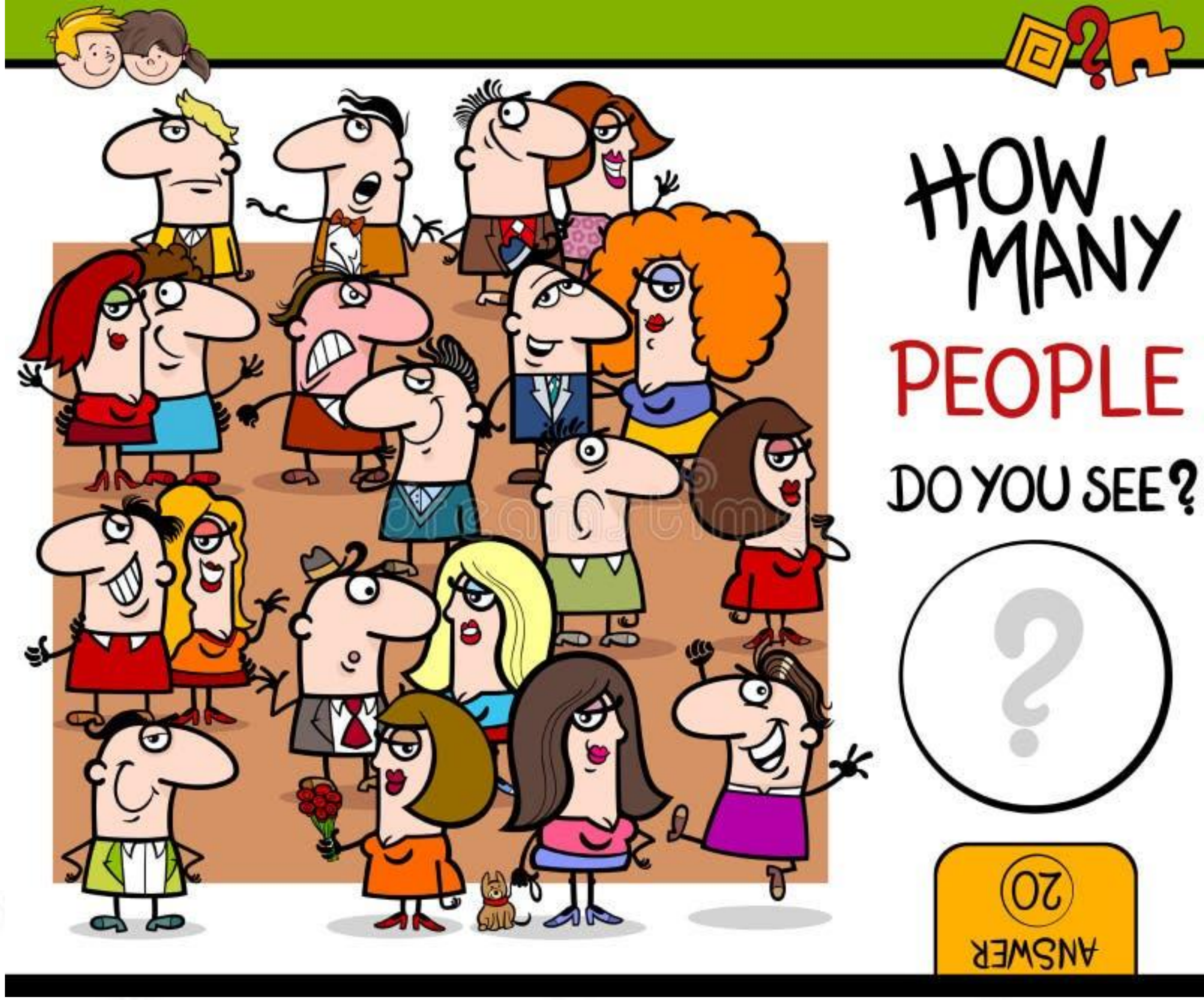
- Smith has good evidence that …
  - Jones is the person who will get the job.
  - Jones has ten coins in their pocket.
- Smith infers from these premises that …
  - The person who will get the job has ten coins in their pocket.

- But unexpectedly, Jones does not get the job. However, Smith actually gets the job and Smith also has tens coins in their pocket.

Knowledge From Falsehood

- I have good evidence that …
  - There are 53 people in the audience.
  - I have 100 handouts.
- I infer from these premises that …
  - I have enough handouts for everyone.

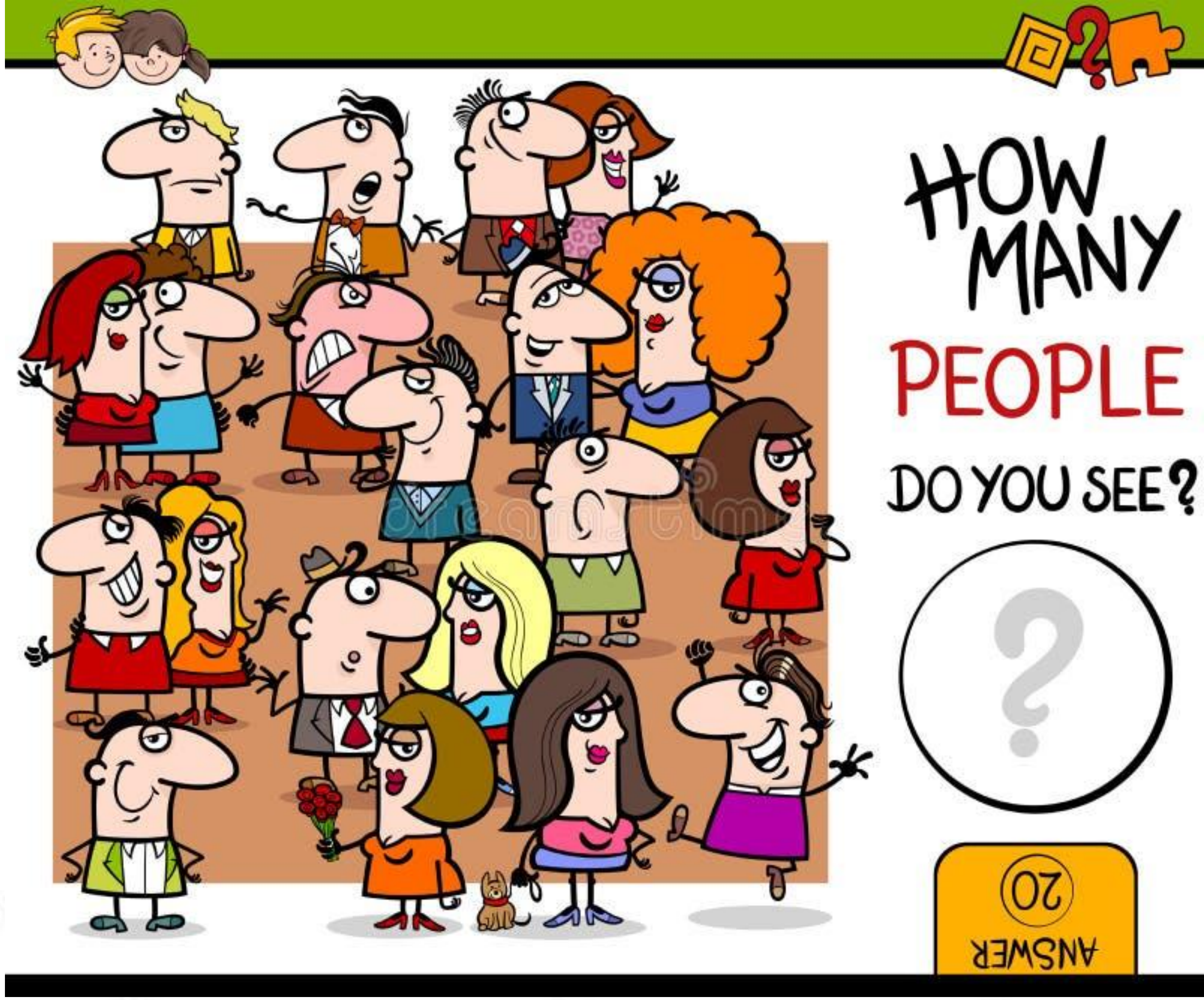
- Unfortunately, *unbeknownst to me*, someone moved seats while I was counting and I miscounted by 1.

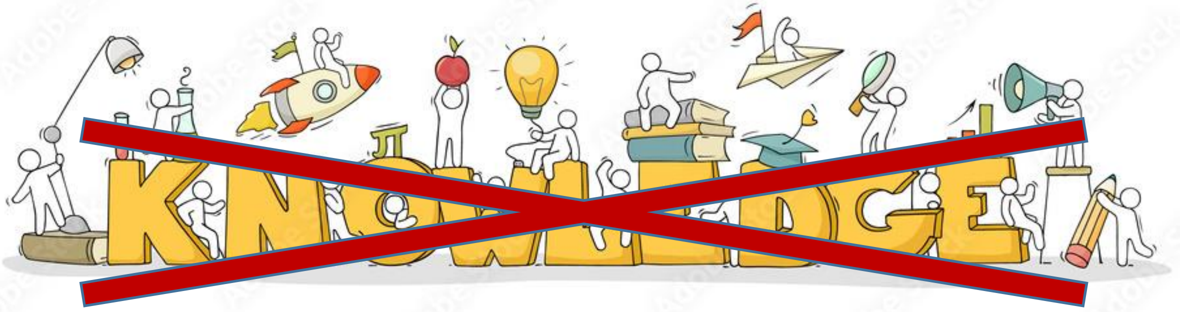


A Variation on the Handout Case

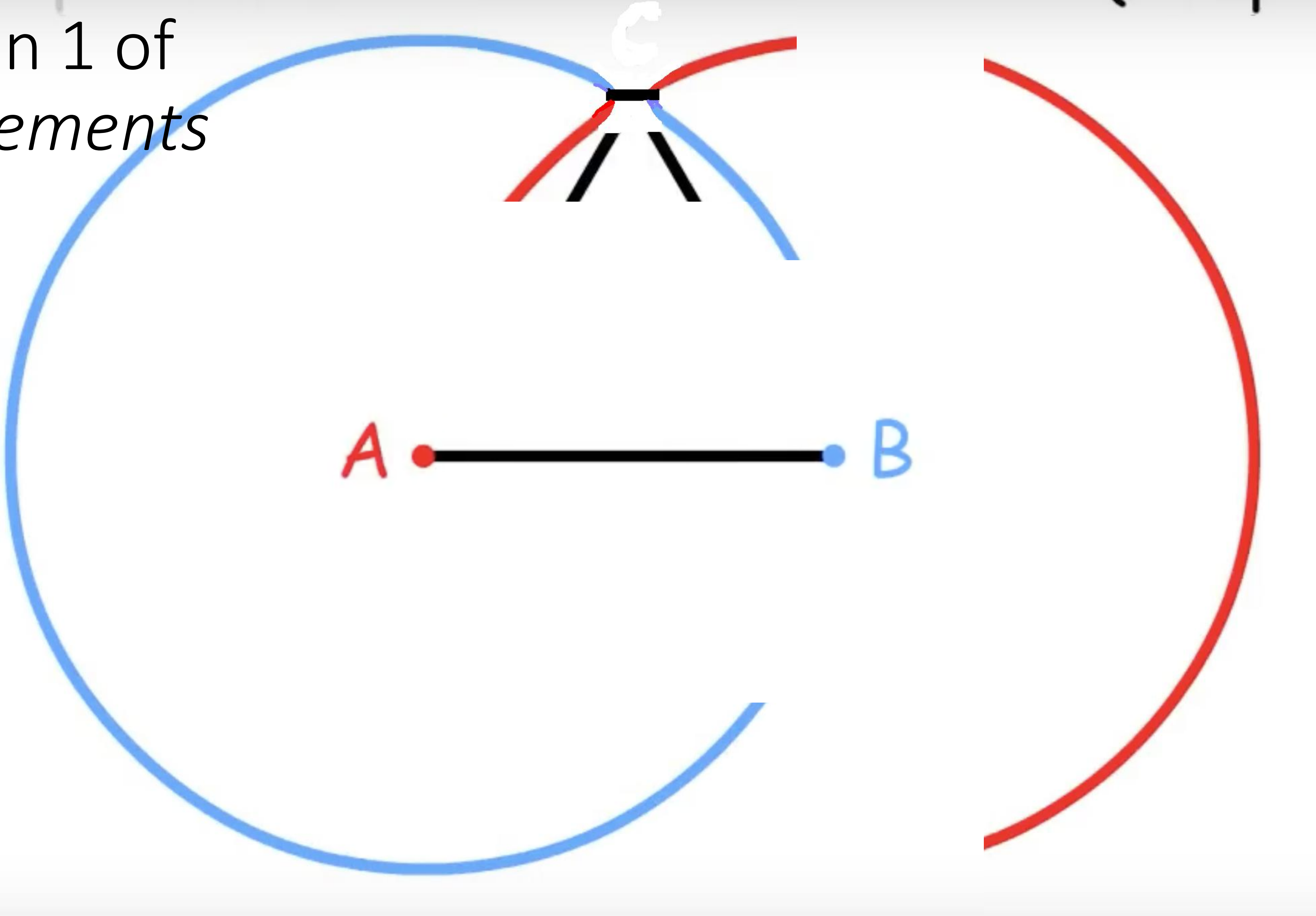
- I have good evidence that …
  - There are 53 people in the audience.
  - I have 100 handouts.
- I infer from these premises that …
  - There are more than 52 people in the audience.

- Unfortunately, *unbeknownst to me*, someone moved seats while I was counting and I miscounted by 1.

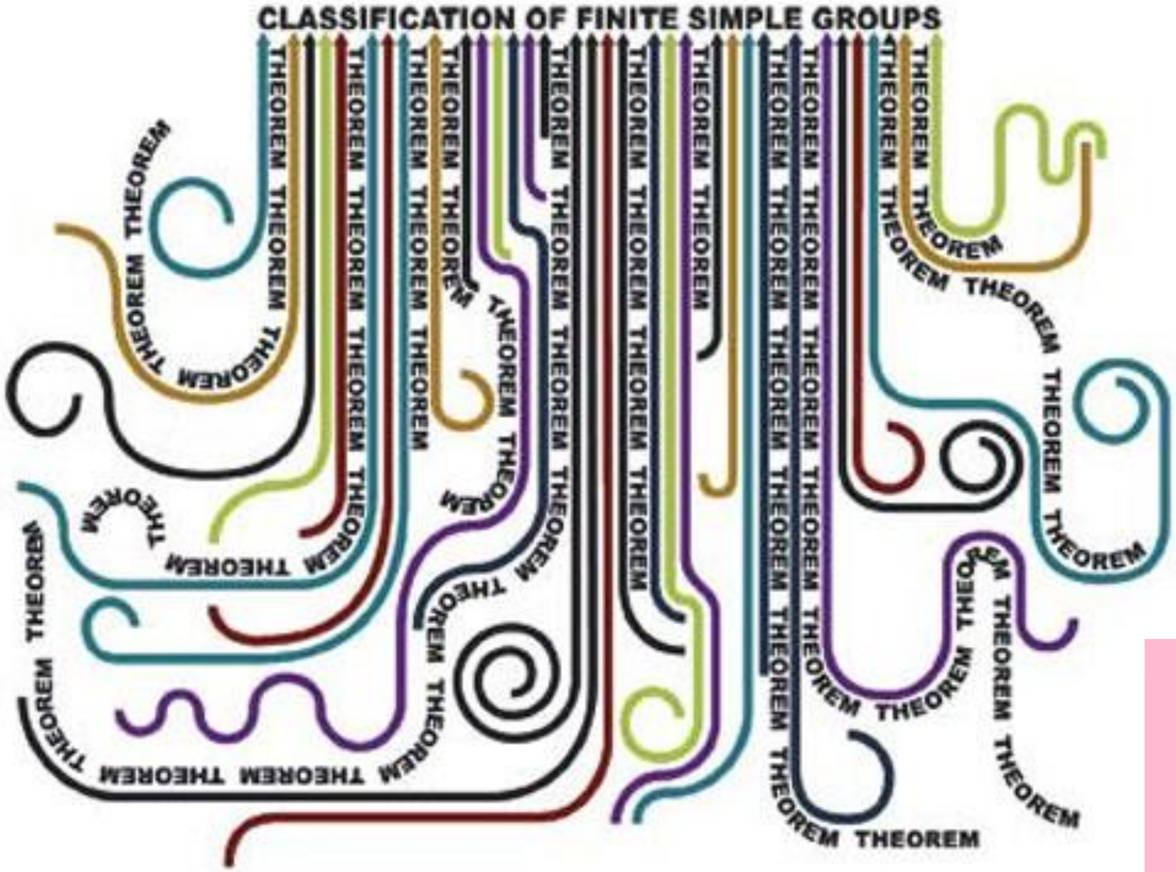


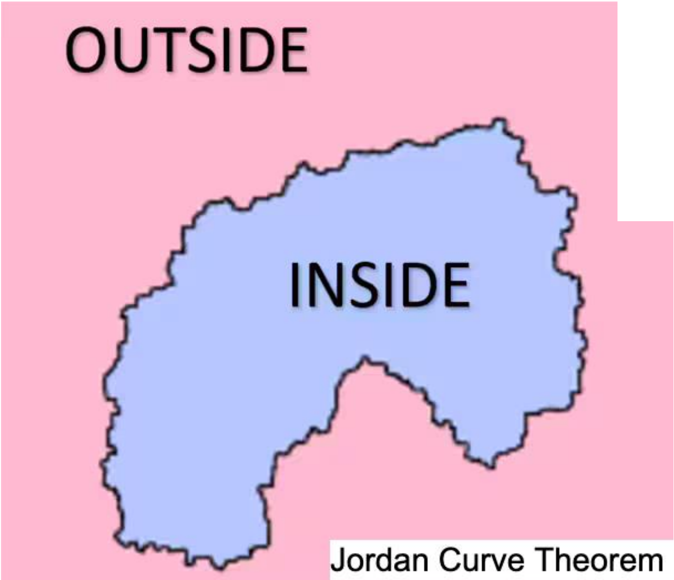


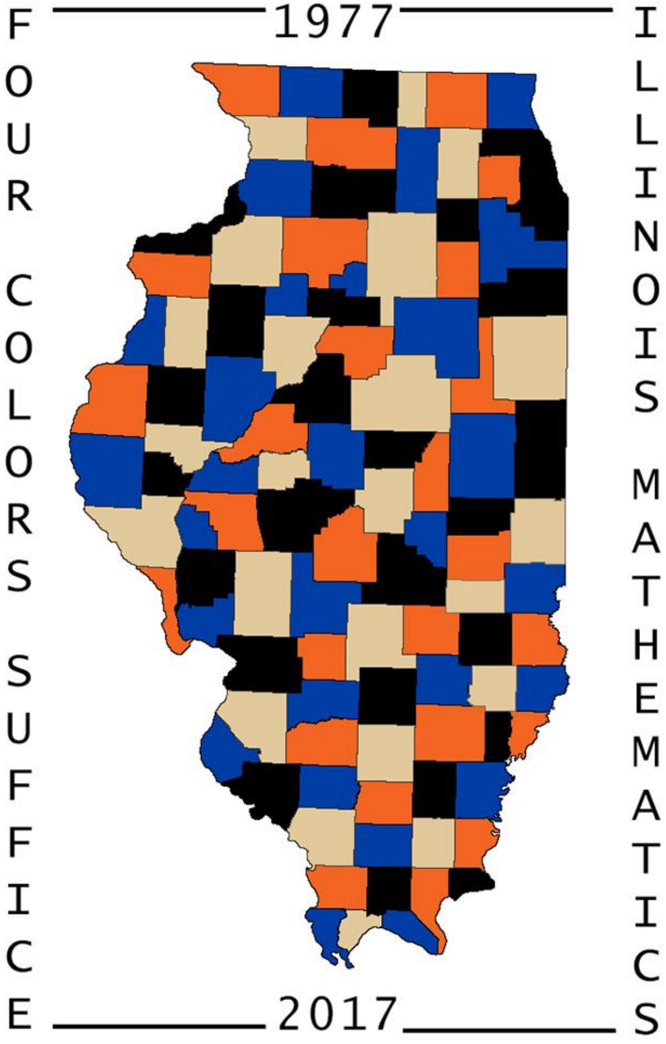
Proposition 1 of Euclid's *Elements*



Many Mathematical Examples

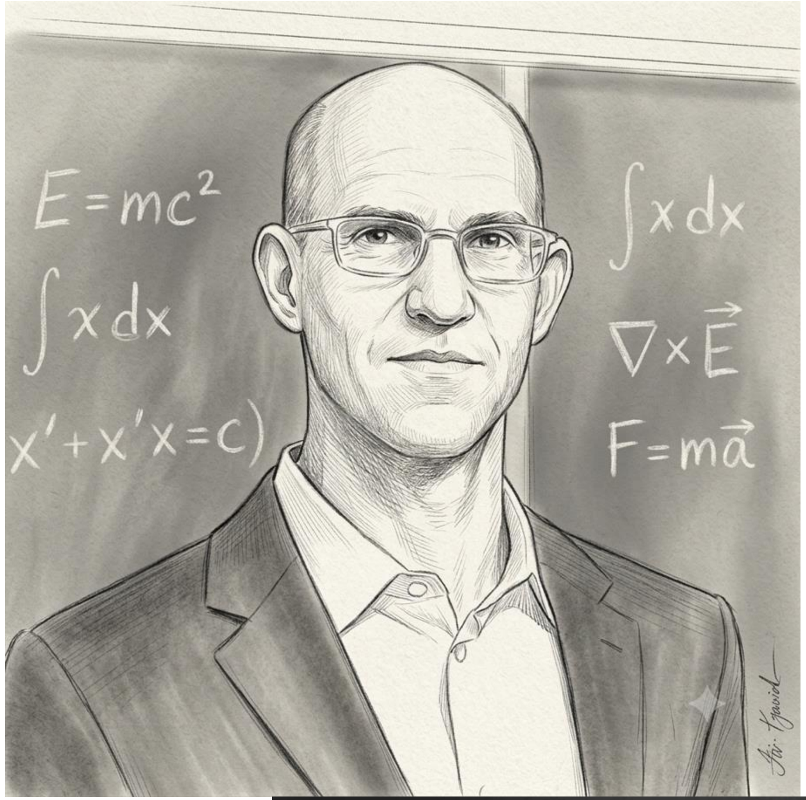


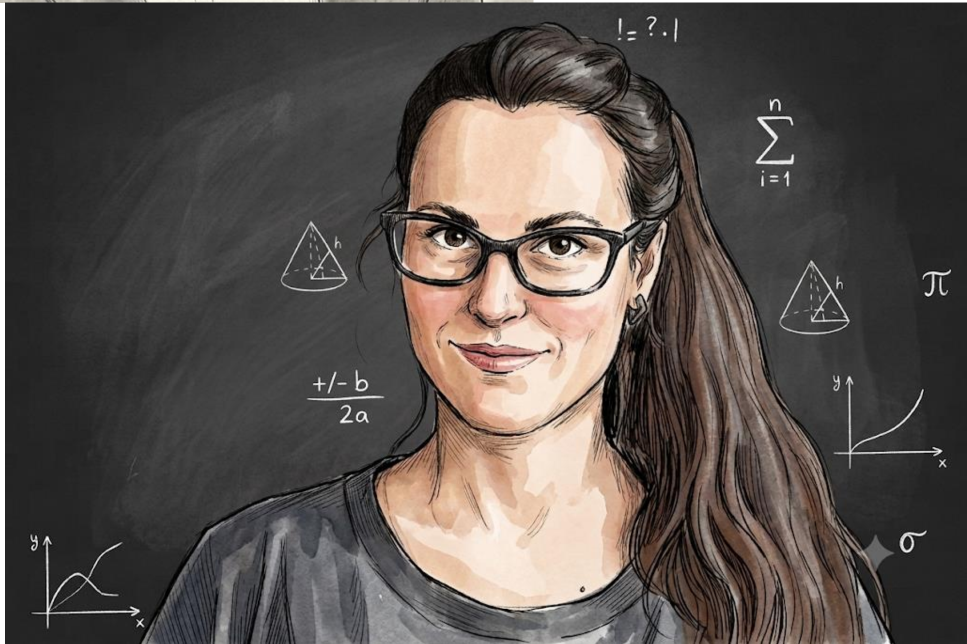




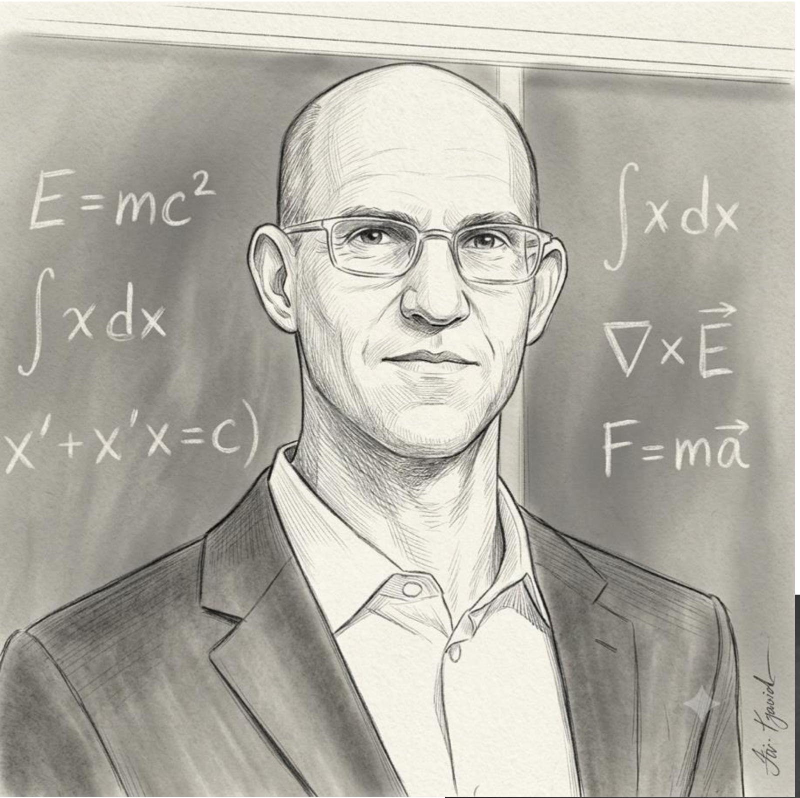
Small Mistakes are Okay

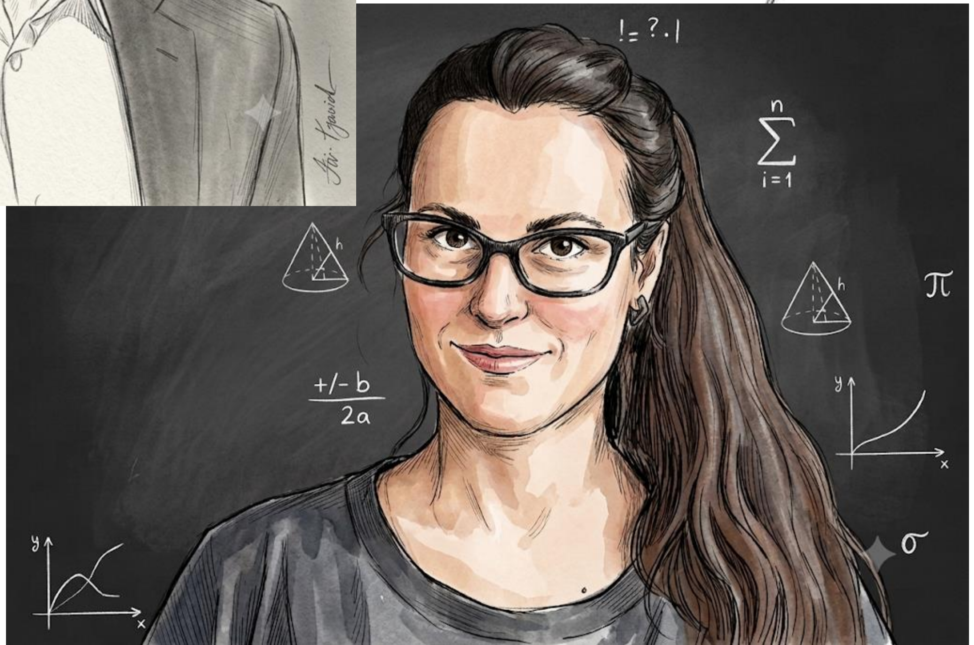
- "An informal proof can have small mistakes and yet still reasonably lead us to believe in the correctness of its conclusion. It is not a contradiction in terms to credit a paper with establishing a certain result while at the same time noting that one of the lemmas in that paper is misstated, if the problem can be fixed and the fix is viewed as minor."
- "Saying that fallacious putative proofs aren't proofs at all cannot mean that putative proofs containing minor mistakes are no proofs because those are ubiquitous in mathematics; banning them would lead to excluding too many of the things that mathematicians are happy to call proofs. That is why *essentially correct* putative proofs are proofs."





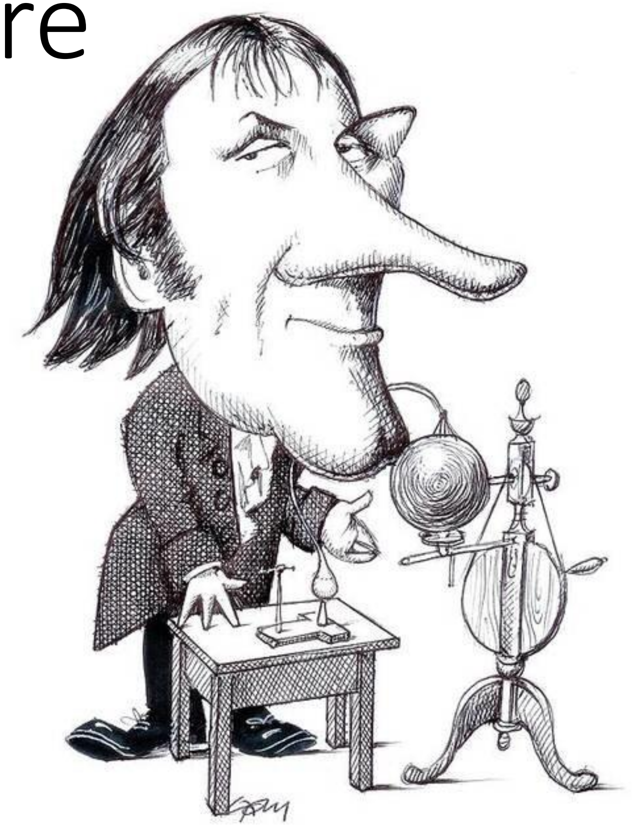
How Do Fugu Yield Knowledge?





- What (if anything) distinguishes Turing 1937 and Perelman 2003 from Kempe 1879 and Wiles 1993?

Credit for Proving the Poincaré Conjecture



- We give Joseph Priestley credit for discovering oxygen.
- But it is not clear that Priestley ***knew*** that he had discovered oxygen.

The Friday the 13th Case

- Jinx has been booked on a flight on the 13th.
- By the law of large numbers, the 13th is equally likely to be any particular day of the week.
- There are seven days in a week.
- Therefore, the probability that Jinx's flight will be on Friday is 1/7.
- Therefore, it is unlikely that Jinx's flight will be on Friday the 13th.

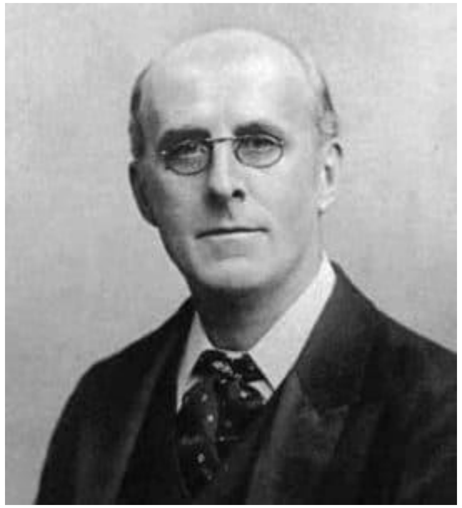
The Gregorian Calendar is Cyclic

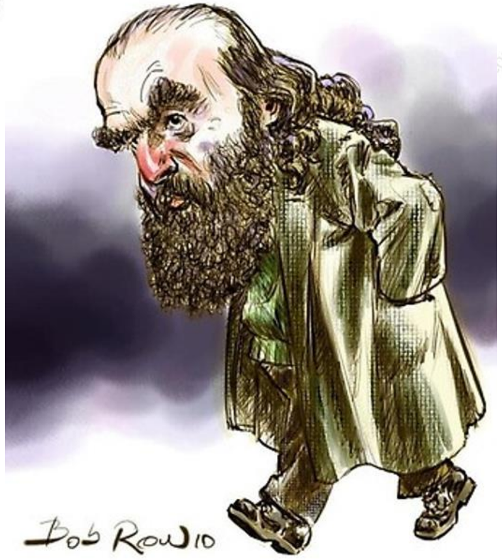
- The Gregorian calendar repeats the same sequence of dates every 400 years.
- The 400-year sequence is 146,097 days long, which is divisible by 7.
- So, the calendar repeats the same sequence of day-of-the-week and date combinations every 400 years.
- So, the very same sequence of Friday the 13th's will occur every 400 years.
- So, in the long run, the percentage of Friday the 13th's will converge to its percentage in this 400-year period.
- Is that percentage exactly 1/7?
- **No**, here's why …
- The 400-year sequence is also 4,800 months long.
- So, it includes 4,800 days which are the 13^{th} day of the month.
- But 4,800 is not divisible by 7.
- Since the number of 13^{th}'s is not divisible by 7, the seven days of the week can't all get the same number of them.
- So, the seven different days of the week ***cannot all be equally likely*** to be the 13^{th}.
- In fact, a 13^{th} is slightly more likely to be a ***Friday*** than any other day of the week (see Brown 1933).

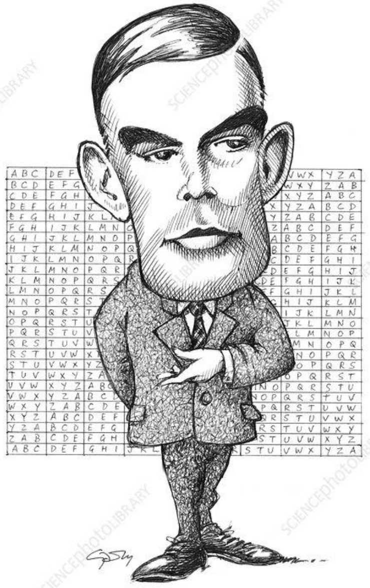
Same Structure as the Handout Case

- Jinx has been booked on a flight on the 13^{th}.
- By the law of large numbers, the 13^{th} is equally likely to be any particular day of the week.
- There are seven days in a week.
- Therefore, the probability that Jinx's flight will be on Friday is 1/7.
- Therefore, it is unlikely that Jinx's flight will be on Friday the 13^{th}.

No New Mathematics

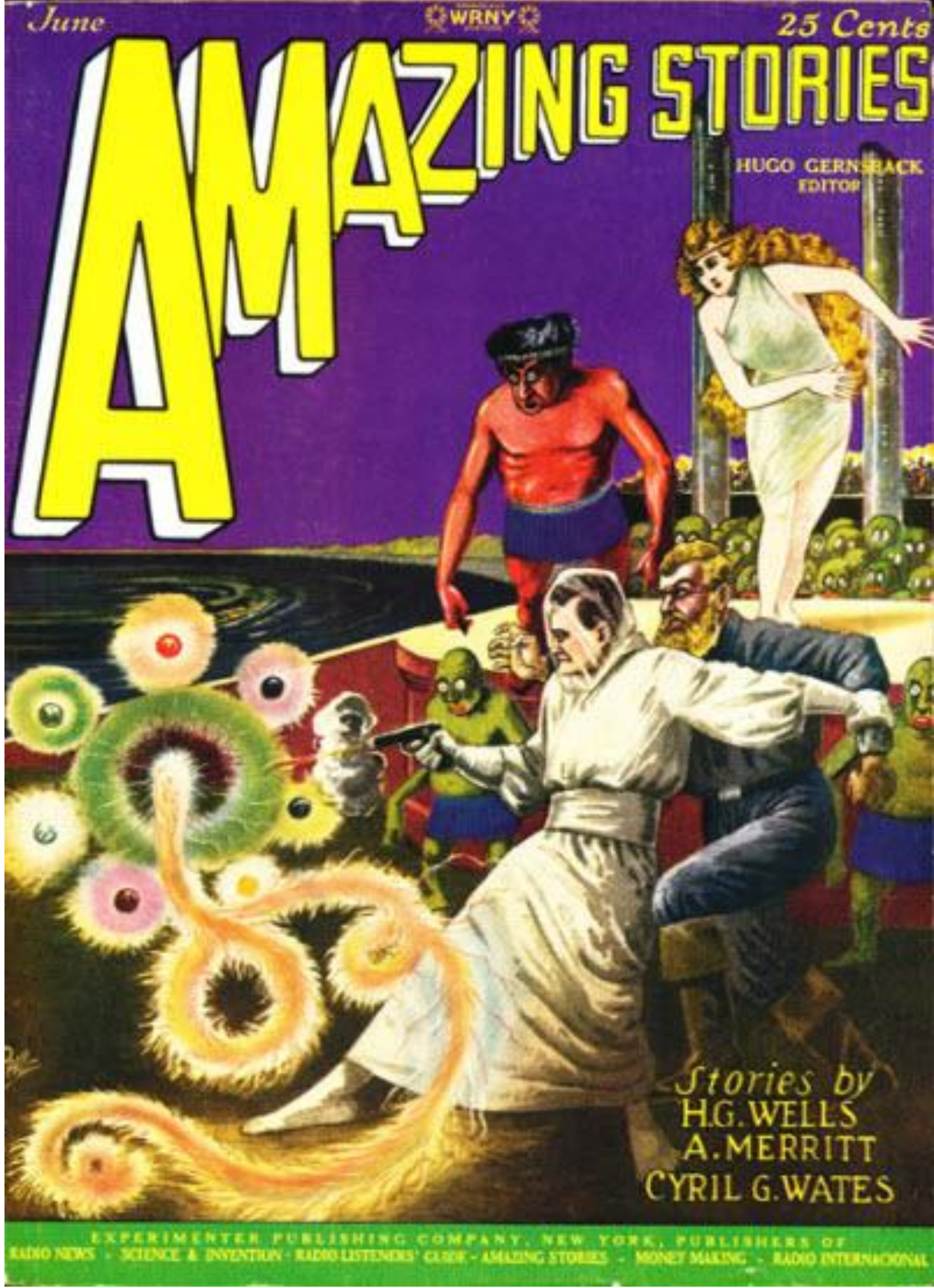






A Simplified Calendar

- “It was New Year’s Day, not January 1st as they had it in the old days, but the extra New Year’s Day that was sandwiched as a separate entity between two years. This new chronological reckoning had been put into use in 1938. The calendar had previously contained twelve months varying in length from twenty-eight to thirty-one days, but with the addition of a new month and the adoption of a uniformity of twenty-eight days for all months and the interpolation of an isolated New Year’s Day, the world’s system of chronology was greatly simplified.”



Mathematical Vision

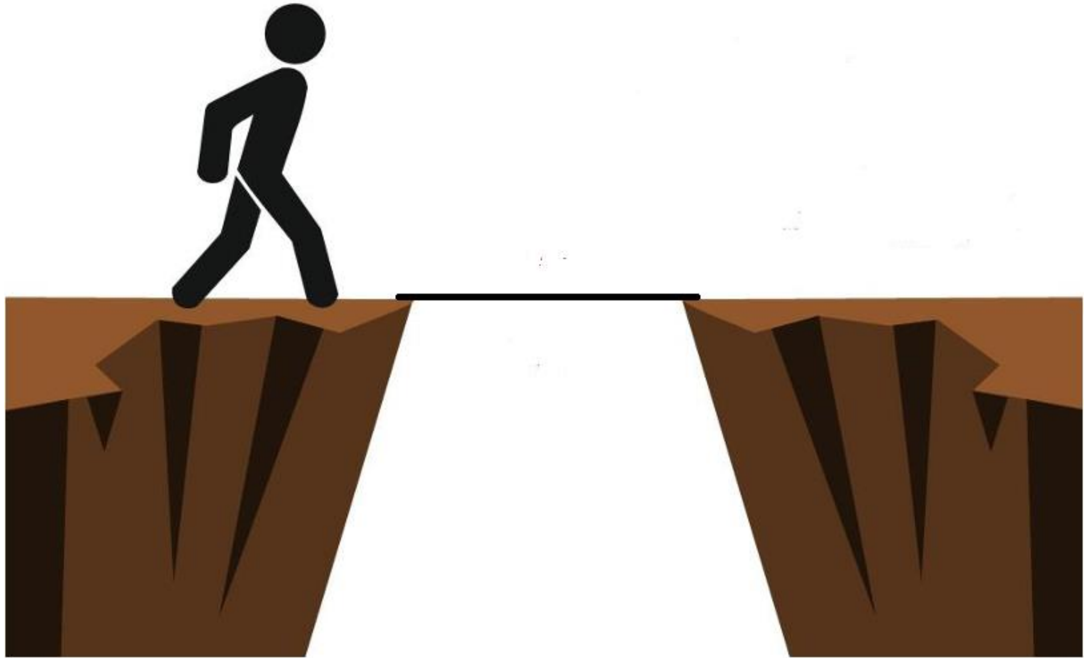
- A mathematician ***knows a proposition*** if they can ***see*** that there is a sequence of steps leading to this proposition—along the same basic lines as the sequence of steps actually proposed—that ***is*** a correct deductive proof (i.e., where each step follows from previous steps by a truth-preserving rule of inference).

The Taking Condition on Inference

- **Making an Inference:**
  - It is not just that believing some premises led you to believe a conclusion.
  - You have to take the premises to be a ***reason*** to believe the conclusion.
  - Thus, you have to ***see*** that the premises support the conclusion.

Fallis on Gaps in Mathematical Proofs

- Inferential Gaps
  - The specific path that you have in mind does not reach the intended destination.
- Enthymematic Gaps
  - You do not describe all of the steps of the specific path that you have in mind.
- Untraversed Gaps
  - You have not tried to follow all of the steps of the specific path that you have in mind.

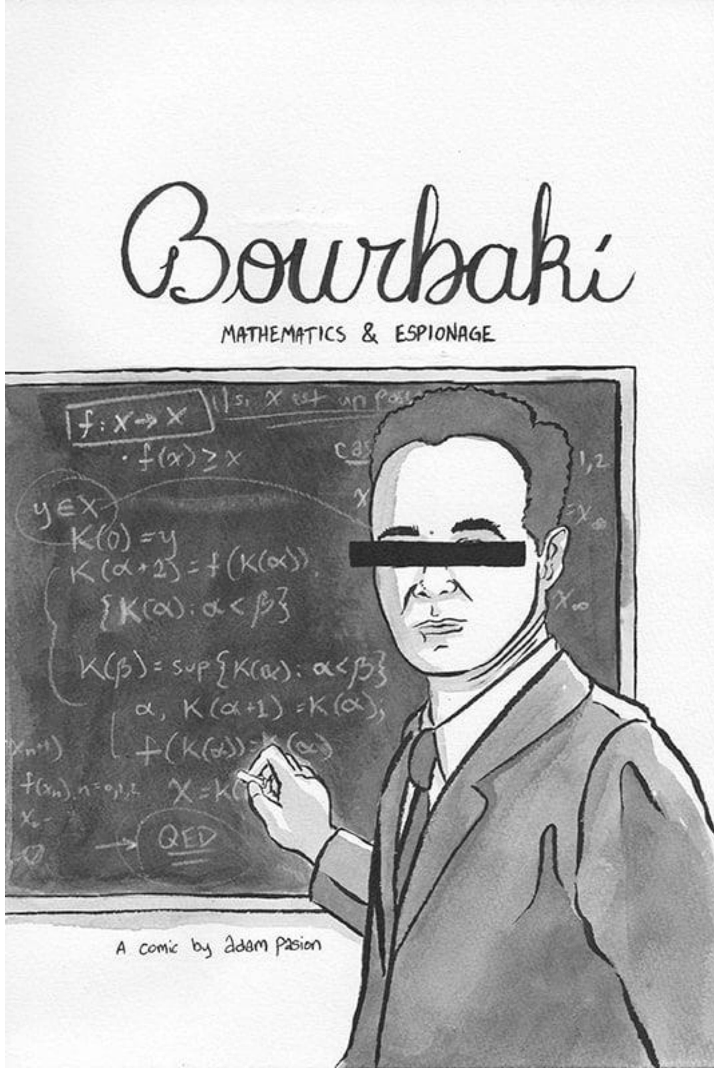


Lamb on Untraversed Gaps

- "A traveler who refuses to pass over a bridge until he has personally tested the soundness of every part of it is not likely to go far; something must be risked, even in mathematics."

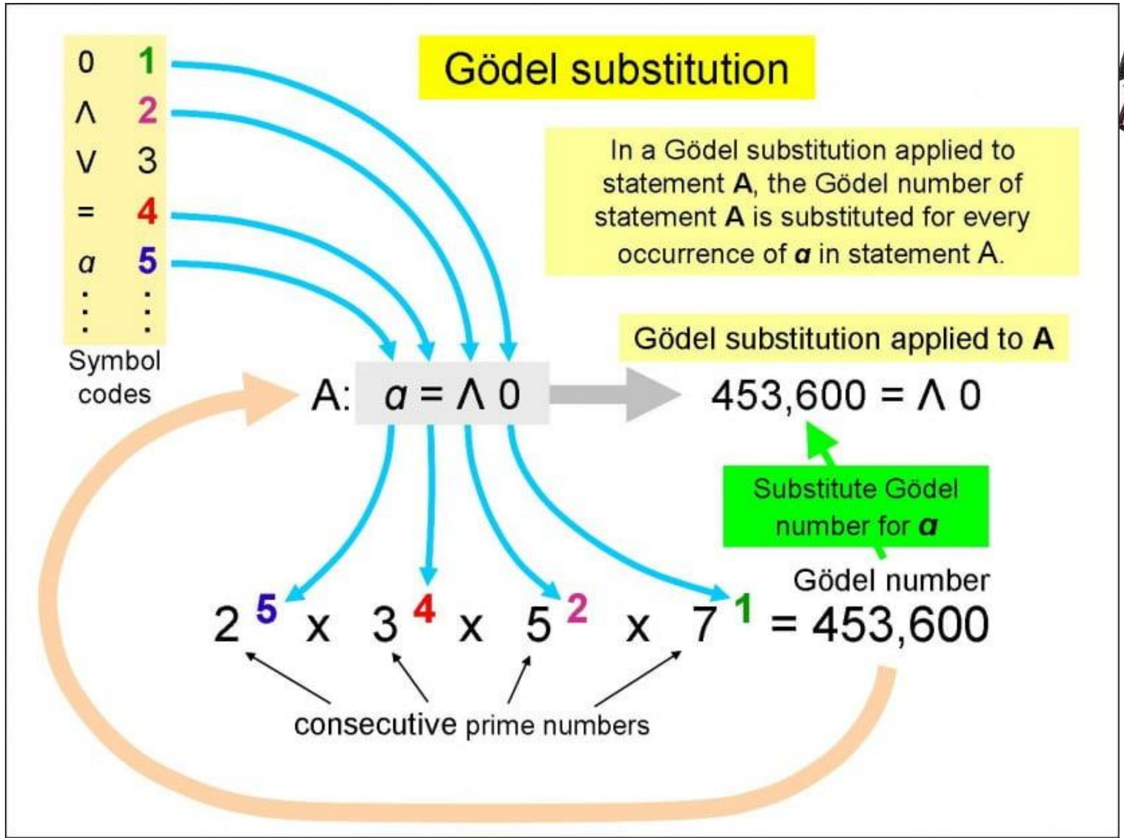
Dieudonné on Mathematical Vision

- "It happens that a proof involves a large number of stages, some of them requiring huge efforts to be set up, while by contrast, the statement 'P implies Q' appears simple, often more or less similar to standard modes of argument. The tendency is therefore to look at it rapidly, often without writing anything down: ***the mathematician, surfeited with details, dispenses with a written proof, replacing it with the ever-recurring phrase 'It is easy to see that . . . '*** " (Dieudonné 1992).



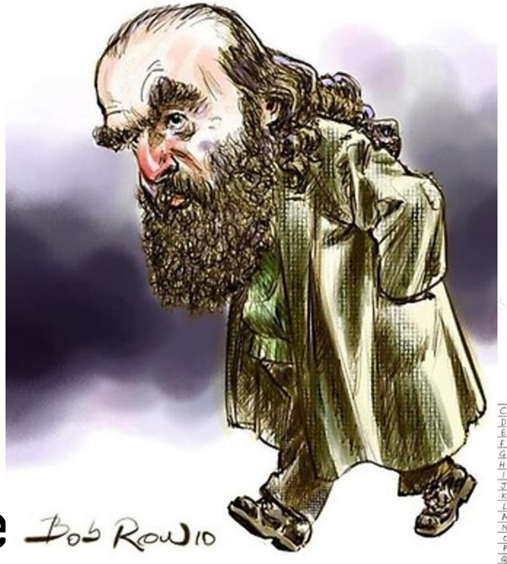
Gödel's Second Incompleteness Theorem

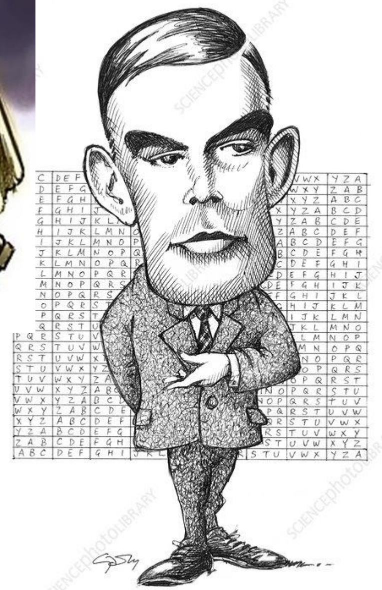
- "The results will be stated and proved in fuller generality in a forthcoming sequel. There too, the mere outline proof we have given of Proposition XI will be presented in detail" (Gödel 1992).



Perelman, Turing, and the Refereeing Process

- “When a referee checks a proof for correctness, she usually does not check every step of the proof … She begins by holding the proof, considered in broad outline, up against the landscape of what she knows. ***She checks whether each subresult of the proof seems reasonable in light of what she knows and, at least for most of the subresults, whether it seems reasonable that this type of result can be proved in this type of way, with this type of tools.*** If so, she will usually not go on to check the subproof line by line … She then turns her focus to the parts that stand out as surprising or suspicious. These are the parts she typically checks line by line” (Andersen 2017).





Thanks!

A Variation on the Friday the 13th Case

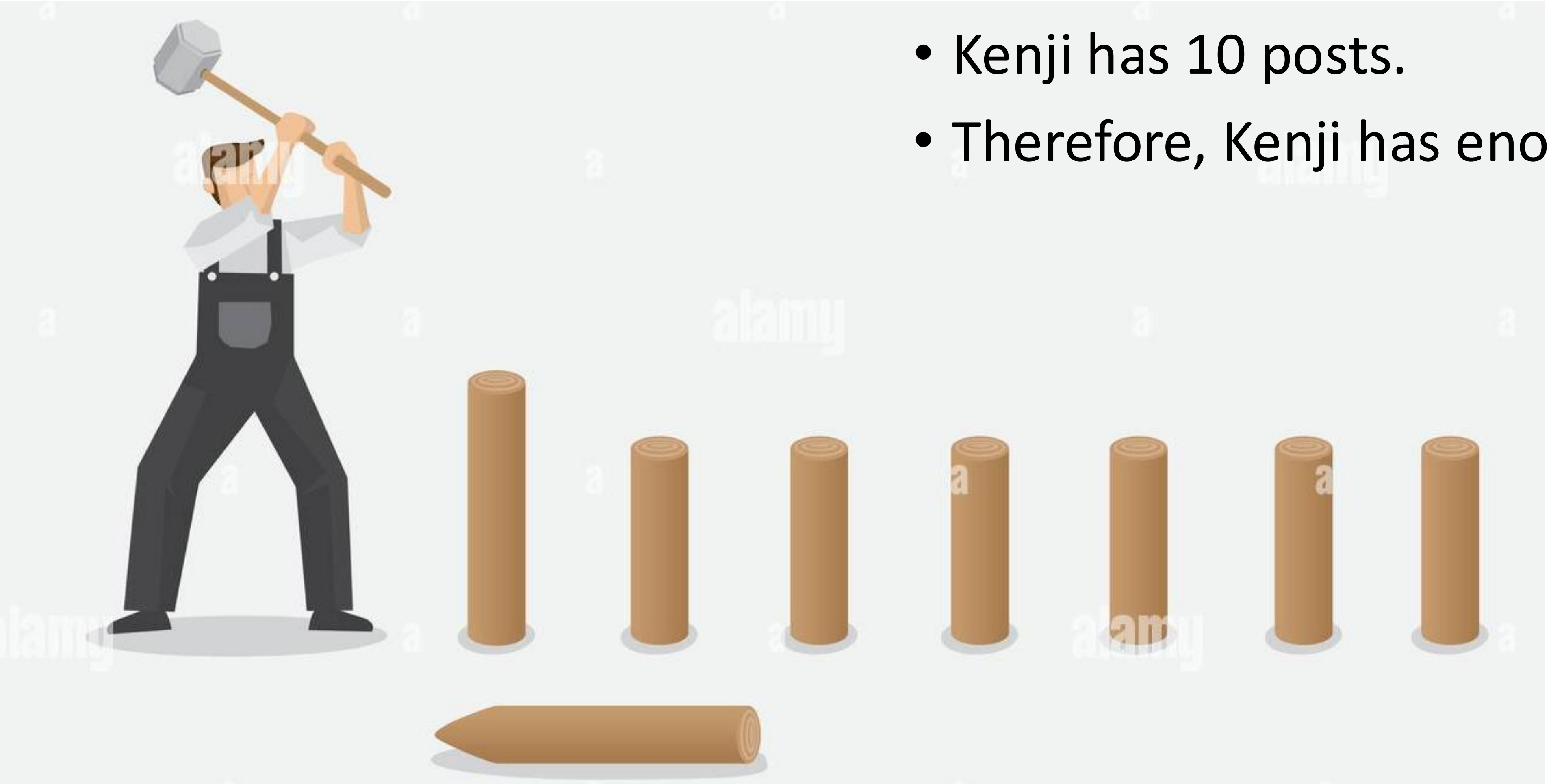
- Jinx has been booked on a flight on the 13th.
- By the law of large numbers, the 13th is equally likely to be any particular day of the week.
- There are seven days in a week.
- Therefore, the probability that Jinx's flight will be on Friday is 1/7.
- Therefore, it is no more likely that Jinx's flight will be on Friday the 13th than that it will be on Monday the 13th.

The Fence Posts Case

- Kenji wants to build a straight fence 30 meters long with posts spaced 3 meters apart.
- Kenji has less than 10 posts.
- Therefore, Kenji does not have enough posts.

A Variation on the Fence Posts Case

- Kenji wants to build a straight fence 30 meters long with posts spaced 3 meters apart.
- Kenji has 10 posts.
- Therefore, Kenji has enough posts.

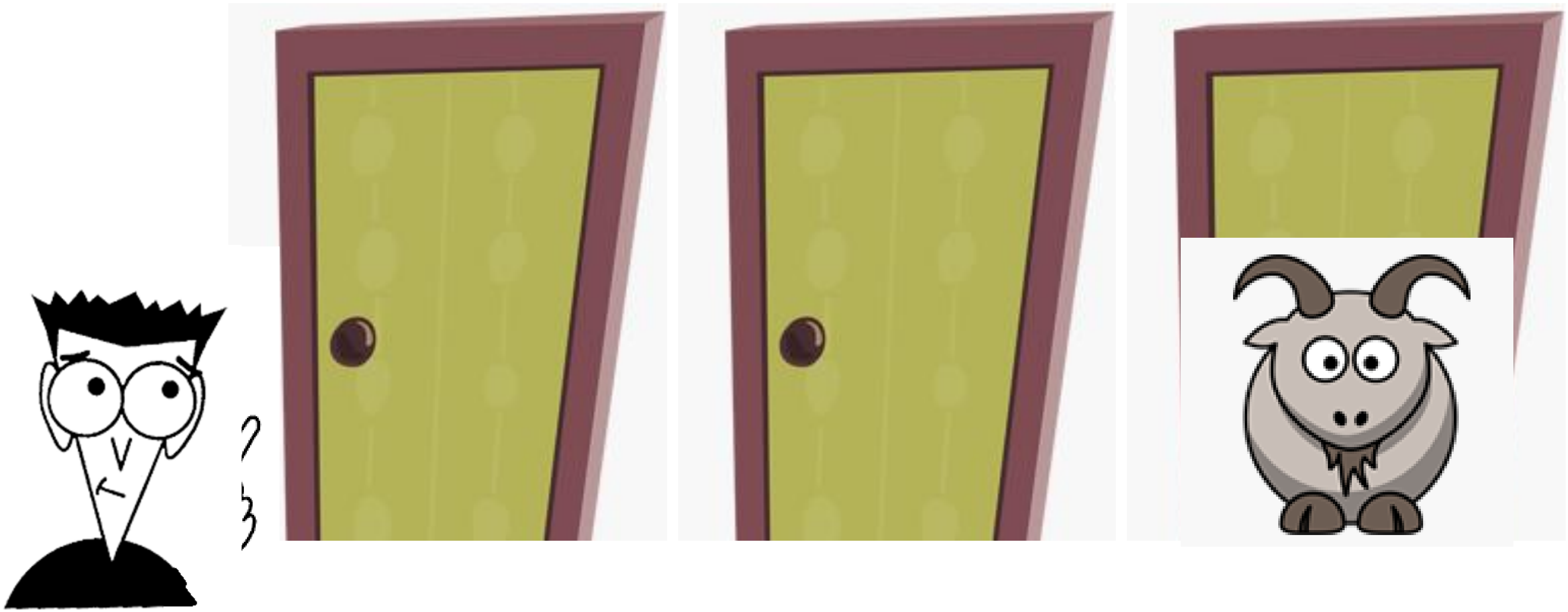


Just Got Lucky

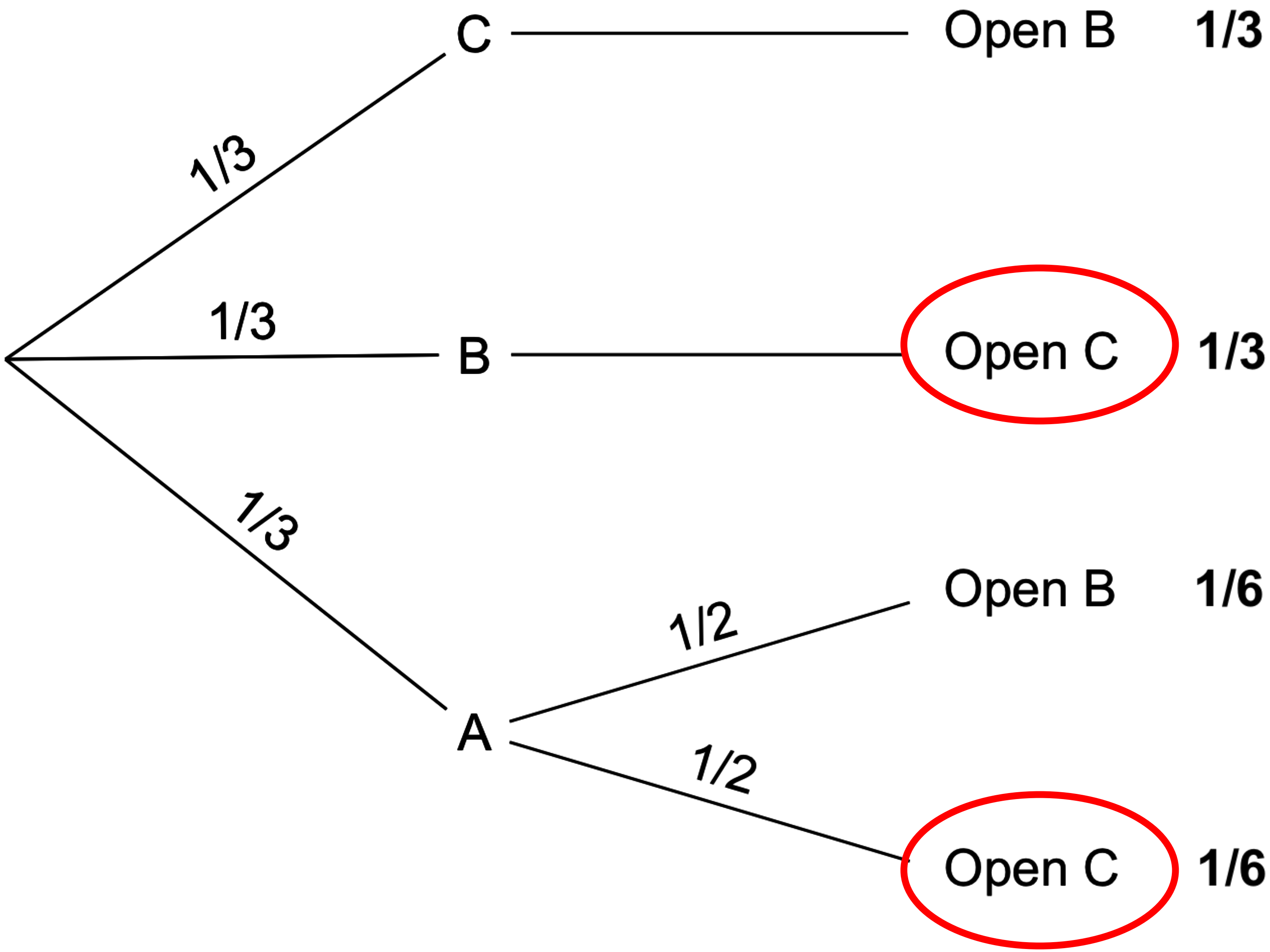
The Monty Hall Problem

- A car is randomly placed behind one of three doors. (Goats are placed behind the other two.)

- You pick a door.
- Monty opens another door to reveal a goat.
- Should you switch to the remaining closed door?



Bayes on the Monty Hall Problem



Sorensen on the Monty Hall Problem

- "Monty's revelation that door C has a goat cannot raise the probability that door A has the prize because you already knew that Monty was going to either reveal B as a loser or reveal C as a loser. … [Thus] … the probability that the prize is behind door B rises to 2/3 because the probability of door A winning is not affected."

The Lazy Monty Variation

- A car is randomly placed behind one of three doors. (Goats are placed behind the other two.)
- The latch on the middle door sticks. So, Monty only opens it when he absolutely has to.
- You pick a door.
- Monty opens the other non-sticky door to reveal a goat.
- Should you switch to the remaining closed door?

Bayes on the Lazy Monty Variation