MathFest 2025
Invited Paper Session
Philosophy of Mathematics: The View from Paradox

Mathematical Paradoxes and the Evolution of Philosophical Commitments

Martin Flashman
Cal Poly Humboldt
flashman@humboldt.edu

Abstract

Throughout history **mathematical paradoxes have been problematic, leading eventually to a growth in understanding of both the nature of mathematical objects and mathematical knowledge**. The resulting changes may be ascribed to the pragmatic acceptance of new commitments for the foundations of mathematics.

In this presentation **I will examine some key paradoxes, and their resolution and acceptance as noncontradictory statements.** I will consider the paradoxes of Zeno, Russell, ~~and Lowenheim-Skolem~~ and how their resolutions were connected to changes in the philosophical foundations of mathematics.

Disclaimer

This only sketches **an outline for an everyperson approach** to paradoxes in the philosophy of mathematics-

It is a sequel borrowing much from my 2010 and 2022 papers-

The Articulation of Mathematics:

An Everyperson Pragmatic/Constructive Approach to The Philosophy of Mathematics and

Choices and Commitment for the Philosophy of Mathematics.

As with most philosophy and "short" presentations-

Expect more questions than answers or specific details.

Thanks!

To the Organizers of this Session for inviting me to present and allowing me so much time for my presentation.

To the audience for attending at this early hour.

If you, as a member of the audience, are disappointed in what follows, I will accept all the blame. ☺

Mathematical Philosophy Framework(s)

The philosophy of mathematics has often taken mathematics as ***a realm of discourse that is fixed***.

The **investigation** of this realm is what working mathematicians take as their task, leading to **results and reports** on

- what they have ascertained and
- the methods used in these investigations.

Communications accompanying these reports and results allow others

- to achieve comparable experiences of understanding or
- to accept the results for further investigations.

An alternative "constructive" view:

- The mathematical realm is **dynamic and changing.**
- The task of _working mathematicians_ involves
  - **a pragmatic effort of developing and relating concepts;**
  - **the articulation of this realm;**
  - **tools useful for analyzing this realm.**
- **This work leads to results and reports on**
  - what they have found useful and
  - the methods used in this process.
- **Communications** accompanying these reports and results allow others
  - to achieve and extend comparable conceptual frameworks;
  - to accept the frameworks for further development.

Paradox

What Is A Paradox?

A ‘paradox’ is a statement and a demonstration that a contradiction or absurd consequence follows from apparently reasonable assumptions about a context. [1]

[1] Huggett, Nick, "Zeno’s Paradoxes", *The Stanford Encyclopedia of Philosophy* (Fall 2024 Edition), Edward N. Zalta & Uri Nodelman (eds.), https://plato.stanford.edu/archives/fall2024/entries/paradox-zeno/.

2 Types of Paradoxes?

A **syntactic** paradox is a contradiction or self-referential statement that arises from the structure or rules in the context of a language or formal system (syntax).

A **semantic** paradox is a paradox that arises from the meaning of words and phrases of a statement in a context (semantics), particularly those related to truth and reference.

The Role of Paradoxes in Mathematics and Philosophy

Throughout history, paradoxes have provided challenges for mathematicians and philosophers to explore a context for both the fundamental nature of (mathematical) **objects- "ontology"** and (mathematical) **knowledge- "epistemology"**.

Epistemology: Definition?

The **theory of knowledge**, especially with regard to its methods for ascertaining validity and their scope.
I.e.,
Epistemology is the investigation of **what distinguishes justified belief from opinion**.

What Is A Paradox? Revised

A ‘paradox’ is a statement and a demonstration that a contradiction or absurd consequence follows from apparently reasonable assumptions ***about the ontology, and/or the epistemology of a philosophical, and/or mathematical*** context.

Serenity and Philosophy

Serenity for Working Mathematicians

- Serenity to accept what we cannot change.

Serenity for Working Mathematicians

- Serenity to accept what we cannot change.
- **Courage to change what we can.**

Serenity for Working Mathematicians

- Serenity to accept what we cannot change.
  - Courage to change what we can.
- **Wisdom to know the difference.**

“Philosophy of Acceptance” for a Working Mathematician

- Some mathematical objects exist.

"Philosophy of Acceptance" for a Working Mathematician

- Some mathematical objects exist.
- Ontological commitments come from personal and common experiences.

"Philosophy of Acceptance" for a Working Mathematician

- Some mathematical objects exist.
- Ontological commitments come from personal and common experiences.
- **Methods for accessing mathematical knowledge are evolving, not static.**

“Philosophy of Acceptance” for a Working Mathematician

- Some mathematical objects exist.
- Ontological commitments come from personal and common experiences.
- Methods for accessing mathematical knowledge are evolving, not static.
- **Epistemological commitments come from personal and common experiences**

"Philosophy of Acceptance" for a Working Mathematician

Differences and conflicts in ontological and epistemological beliefs and viewpoints are **evidence for not looking for a universal and eternal foundation for either.**

Response for Everyperson

Mathematics **evolves** in a **dynamic** process of **articulation**.

Response for Everyperson

Mathematics **evolves** in a **dynamic** process of **articulation**.

The result of work in mathematics is **an inter-related web or fabric of information-data and concepts.**

Response for Everyperson

Mathematics **evolves** in a **dynamic** process of **articulation**.

The result of work in mathematics is **an inter-related web or fabric of information-data and concepts.**

What survives in mathematics is a result of a **pragmatic standard founded on (scientific) empiricism and consistency.**

Zeno's (Motion) Paradoxes

Zeno's (Motion) Paradoxes

1. Dichotomy Paradox:

To reach a destination, one must first cover half the distance.

To cover the initial (or remaining) half, one must cover half of that, and so on.

This process, repeated infinitely, suggests that ***movement is impossible.***

Zeno's (Motion) Paradoxes

1. Dichotomy Paradox:
To reach a destination, one must first cover half the distance.
To cover the initial (or remaining) half, one must cover half of that, and so on.

This process, repeated infinitely, suggests that movement is impossible.

2. Achilles and the Tortoise:

If a slower runner (like a tortoise) is given a head start, the faster runner (Achilles) can never overtake them.
By the time Achilles reaches the tortoise's starting point, the tortoise will have moved a small distance further, and so on. This process, repeated infinitely, suggests that *Achilles will never pass the Tortoise.*

Zeno's (Motion) Paradoxes

1. Dichotomy Paradox:
To reach a destination, one must first cover half the distance.
To cover the initial (or remaining) half, one must cover half of that, and so on.
This process, repeated infinitely, suggests that movement is impossible.

2. Achilles and the Tortoise:
If a slower runner (like a tortoise) is given a head start,
the faster runner (Achilles) can never overtake them.
By the time Achilles reaches the tortoise's starting point,
the tortoise will have moved a small distance further, and so on.
This process, repeated infinitely, suggests that Achilles will never pass the Tortoise.

3. The Arrow Paradox:

At any given instant of time, an arrow in flight occupies a specific position in space.
If an instant is indivisible, the arrow cannot be moving during that instant,
and since motion is composed of a series of instants, it cannot be moving at all.

Zeno's (Motion) Paradoxes

1. Dichotomy Paradox:
To reach a destination, one must first cover half the distance.
To cover the initial (or remaining) half, one must cover half of that, and so on.
This process, repeated infinitely, suggests that movement is impossible.

2. Achilles and the Tortoise:
If a slower runner (like a tortoise) is given a head start,
the faster runner (Achilles) can never overtake them.
By the time Achilles reaches the tortoise's starting point,
the tortoise will have moved a small distance further, and so on.

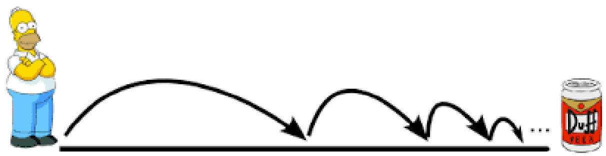
3. The Arrow Paradox:
At any given instant of time, an arrow in flight occupies a specific position in space.
If an instant is indivisible, the arrow cannot be moving during that instant,
and since motion is composed of a series of instants, it cannot be moving at all.

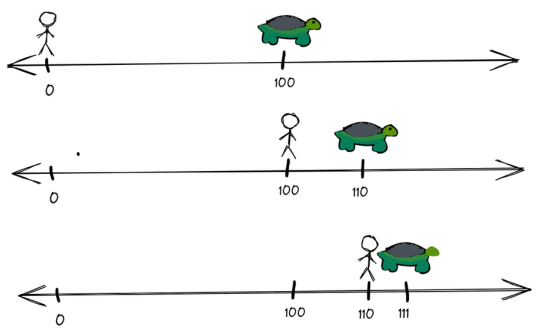
4. The Stadium Paradox:

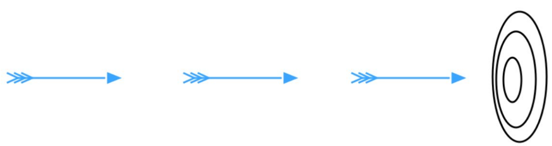
This paradox involves three rows of objects, one stationary, the other two moving at the same speed in opposite direction and moving past the other two.

It questions how *objects can move at different speeds within the same space and time*.

Zeno's (Motion) Paradoxes







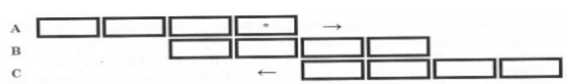


Figure 1: Initial position

Imagine that the A's move to the right, the B's stay stationary, and the C's move to the left, until the arrangement in Figure 2 is produced:

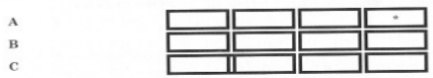


Figure 2: Final position

1. Dichotomy Paradox:
To reach a destination, one must first cover half the distance.

To cover the initial (or remaining) half, one must cover half of that, and so on.

This process, repeated infinitely, suggests that movement is impossible. … ***movement is impossible.***

2. Achilles and the Tortoise:
If a slower runner (like a tortoise) is given a head start,

the faster runner (Achilles) can never overtake them.

By the time Achilles reaches the tortoise's starting point,

the tortoise will have moved a small distance further, and so on. …. ***Achilles will never pass the Tortoise.***

3. The Arrow Paradox:
At any given instant of time, an arrow in flight occupies a specific position in space.

If an instant is indivisible, the arrow cannot be moving during that instant,

and since motion is composed of a series of instants, it ***cannot be moving at all.***

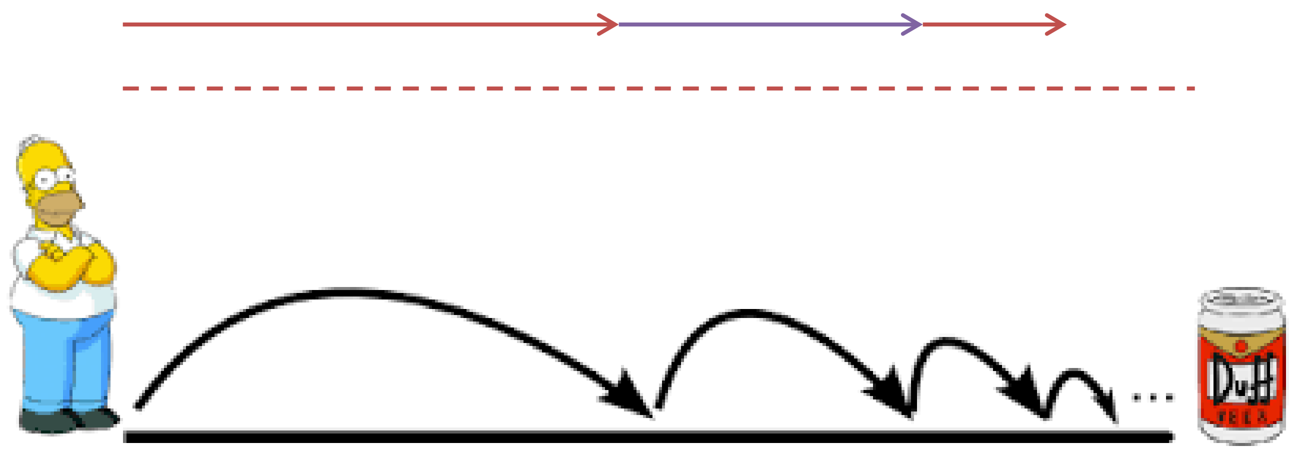
4. The Stadium Paradox:

This paradox involves three rows of objects, one stationary, the other two moving at the same speed in opposite direction and moving past the other two.

It questions how ***objects can move at different speeds within the same space and time***.

Focus on the Dichotomy

The **Dichotomy Paradox** argues that motion is impossible because an object must first traverse half the distance to its destination, then half of the remaining distance, and so on, requiring an infinite number of steps to reach the end. This leads to the conclusion that ***motion cannot be completed in a finite number of steps.***



Questions for Discussion

How would you explain the Dichotomy paradox
-to a class of non-math majors?
-to a class of science majors?
-to a calculus class?
-to an analysis class?

What makes a response to a paradox **satisfactory**?

Questions for Discussion

How would you explain the Dichotomy paradox
-to a class of non-math majors?
-to a class of science majors?
-to a calculus class?
-to an analysis class?

What makes a response to a paradox **satisfactory**?

Questions for Discussion

What is the context?

The context depends on the historical period.

What are the objects in Zeno's Paradox?

- Distances/steps travelled with the motion.
- An interval of time during which the motion happened.
- A fractional measure of distances.

Questions for Discussion

1. Does the paradox depend on the nature of the objects? (**Ontology**)
2. Does the paradox dependent on **context**? physics/geometry
3. Does the paradox depend on how the paradox is explained? (**Epistemology**)

4. What makes a response to a paradox **satisfactory**? Meta…?

Questions for Discussion…

1. Does the paradox depend on the nature of the objects? (Ontology)

Distances: Measurements in geometry.
Points and line segments.
Physical? Abstract? Formal?

Time : Physical measurement by physical change? Abstract? Formal?

Common Fractions:
Units, counting (natural) numbers
duplication/multiplication

Infinity: Potential/Actual (Aristotle)
Physical? Abstract? Formal?

Russell's Paradox

Background:
Gottlob Frege attempted a 2-volume work building mathematics from a foundation of logic using a naïve version of set theory (1893 & 1903).
It allowed **defining sets by any property/predicate-based statement.**

While reviewing the second volume of this work, Bertrand Russell produced **an apparent contradiction** in Frege's formal system.

Russell's Paradox

Russell's Paradox (1902):
The context is **naïve set theory.**
Let P(a,b) mean a is an element of b.
Let **R == {x : ~P(x,x) is true}**
According to Frege's formal system, **R is a well-defined set.**
Suppose P(R,R) is true. This means R is an element of R.
Thus, by definition of R, ~P(R,R) is true, and P(R,R) is false.
Thus P(R,R) cannot be true.
Hence ~P(R,R) is true, and by definition, R is an element of R,
which means P(R,R) is true.

Consequence:

<u>Either</u> P(R,R) is not a proposition
<u>or</u> the set R is not well-defined.

Questions for Discussion

How would you explain Russell's paradox
-to a class of non-math majors?
-to a class of science majors?
-to a calculus class?
-to an analysis class?

What makes a response to a paradox **<u>satisfactory</u>**?

Questions for Discussion

What is the context?

The context depends on the historical period.

What are the objects in Russell's Paradox?

- Sets.
- Relation: "a is an element of b".
- Statements/propositions.

Questions for Discussion

1. Does the paradox depend on the nature of the objects? (**Ontology**)
2. Is the paradox independent of **context**? Set Theory and Logic
3. Does the paradox depend on how the paradox is explained? (**Epistemology**)

4. What makes a response to a paradox **satisfactory**? Meta…?

Questions for Discussion

1. Does the paradox depend on the nature of the objects? (Ontology)

- **Sets**: A Universe of Sets – Physical? Abstract? Formal? Infinite?
- **Relation**: "a is an element of b". What does **"element of" mean**?
- **Statements/propositions**. Physical? Abstract? Formal? Infinite?

Standards for resolving paradoxes

(Proposed by Lori Balster in Zeno's First Paradox of Motion: A Cartesian Perspective, Austegung, Vol. 25 ,No. 2., pp.113-136.)

In order for a solution to the Paradox to be deemed satisfactory, it must:

1) Be clear and distinct. That is, there must be no part of the solution that cannot be clearly perceived with mind's eye.

2) Reconcile our mathematical and physical perceptions. (Context dependent?)

3) Not alter concepts such as infinity, time, distance, or set in an attempt to resolve the paradox.

What resolves a paradox?

Difficulties arise partly in response to the evolution in our understanding in context of what mathematical rigor demands [epistemology?]:
Do solutions depend on what standards of rigor are applied?
A paradox may start with a context of common-sense formulations but then evolves to a resolution over time as the mathematical context develops further.

History?

Presentation ended here. ☺

Infinity and Sets in History?

Plato, Aristotle : a time and geometric continuum?
Euclid, Archimedes: Constructible "numbers". Square Roots.
Ratios: Continued proportions. Cube Roots?
Circumference: Diameter of Circle.
Approximation of "pi".
Method of exhaustion, and approximations.
Incommensurable magnitudes.

Infinity and Sets in History?

Stevin, Napier: Decimal arithmetic
Definition of logarithms: arithmetic & geometric sequences.
Descartes: Arithmetization of Geometry.
Numerical Continuum? Algebraic Real Numbers

Infinity and Sets in History?

Newton, Leibniz: Naïve Limits, infinitesimals
Cauchy, Weierstrass, Dedekind, Cantor:
Limits, Continuum, Real Numbers, Infinite Sets
Gauss, Riemann, Lobachevsky, Bolyai, Kant:
Non-Euclidean Geometry

Infinity and Sets in History?

Frege, Hilbert, Russell, Poincare, Brower, Robinson:
"Foundations for Mathematics"

Godel, Cohen: Set theory and Continuum Hypothesis

Quine, Putnam: "Meta Foundations for Mathematics"

Response for Everyperson

Mathematics **evolves** in a **dynamic** process of **articulation**.

The result of work in mathematics is **an inter-related web or fabric of information-data and concepts.**

What survives in mathematics is a result of a **pragmatic standard founded on (scientific) empiricism and consistency.**

"Philosophy of Acceptance" for a Working Mathematician

Differences and conflicts in ontological and epistemological beliefs and viewpoints are **evidence for not looking for a universal and eternal foundation for either.**

Thanks
The End!

☺

Comments? Questions?

flashman@humboldt.edu
http://www.humboldt.edu/~mef2