"Surveyability" in Hilbert, Wittgenstein and Turing

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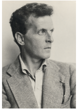
Cast of Characters


Hilbert 1862-1943


Wittgenstein 1889-1951


Turing 1912-1954

Surveyability

Our metaphor is the *surveyability* of proof.

This requires clarification.

1. In Hilbert 1920- we read of the *Überblickbarkeit*, *Übersehbarkeit* and *Übersichtlichkeit* of proof.
2. Wittgenstein's *Blue Book* 1933-4 asks for a *survey* of the concept "humans operating mechanically with signs".
3. Turing's 1936 takes as fundamental the idea of symbols being able to be "taken in at a glance" by (human) computors.
4. Wittgenstein 1937-1944 investigates the "surveyability" of proof in logic and math vs. in philosophy.
5. Turing 1939-1944 emphasizes the development of mathematical "phraseology" for the sake of ... surveyability.

Is “surveyability” a requirement? In what sense?

1. Many proofs and sentences are very difficult to “hold in mind” or “survey at a glance”. We cannot mean a *visual* or a *psychological* criterion.
2. Are computer proofs/models ways of increasing “surveyability”? – Yes and No. Articulation and “phraseology” at the right level are crucial.
3. Ultimately Hilbert, Wittgenstein and Turing have in mind something everyday: what *can be done*, i.e., human communication without disagreement.

Leitmotifs

1. Psychology does not govern the foundations of logic or mathematics, which are not beholden to any particular theory of mind. "No matter, never mind" – Georg Kreisel.
2. The limitative results of Gödel, Post, Church, and Turing do not turn on or entail any particular theory of mind, contra Post and Gödel's reading of Turing, and in favor of Hilbert, Wittgenstein and Turing.
3. We can see this by revisiting Turing's analysis of "formal system of logic", and emphasizing how it is rooted in anti-psychologism about logic inherited from Hilbert and Wittgenstein, as well as certain foundational ideas about simplicity, objectivity, and everyday human "phraseology".

Turing Re-Read

- Turing's philosophical attitude has been distorted by controversies in recent philosophy of mind (Putnam): computationalist and behaviorist reductionisms, functionalism, AI, and the "the singularity", in which machines will inevitably become the primary drivers of cultural change and creativity.
- Turing was neither a behaviorist nor a reductive mental mechanist. Foundations of logic and mathematics, not philosophy of mind, was central for his work.
- The *social* matter of intersubjective communicability was crucial in Turing's philosophy of mathematics (Hilbert, Wittgenstein).

Turing Re-Read

- Turing focussed on taking what we *say* and *do* with words seriously, and on the *limits* of formal methods, not only their power.

- Everyday language, including our "typings" of objects as they occur naturally in science and everyday life, are an evolving framework or technology. Turing stressed human conversation, "phraseology", and "common sense", as foundational. In this sense he was a Cambridge philosopher of his time.

Hilbert 1920, on Proof Theory (my emphasis)

... [P]roof procedures become **completely surveyable** [*Überblickbar*].

. . . the figures we take as objects must be **completely surveyable** and only discrete determinations are to be considered for them. It is only under these conditions that our claims and considerations have the same reliability and evidence as in intuitive number theory (1920, in Sieg 1999, 23,30)

[A] formalized proof, like a numeral, is a concrete and surveyable object. It **can be communicated from beginning to end** (1925, 383)

Subitizing vs. the 'Beginnings' of Mathematics

It is too difficult for a human to take in “at a glance”, i.e., without arranging, counting or labeling, the difference between:

$$||||||||||||\quad \text{and} \quad |||||||||||$$

- Our “number sense”, i.e., “subitizing”, gives out very quickly in the stroke notation.
- It gives out later in the decimal notation.
- Kripke (1992, from Wittgenstein) The first integers in decimal notation serve as *buckstoppers*: they can be taken in, without dispute.
- (Hilbert, Wittgenstein, Turing): there must *be* buckstoppers, this is a representational necessity of *logic and mathematics*, not merely a feature of the human mind or certain specific notations.

Wittgenstein, *Investigations* II xi, §§341, 343

A dispute may arise over the correct result of a calculation (say, of a rather long addition). But such disputes are rare and of short duration. They can be decided, as we say, "with certainty". Mathematicians don't in general quarrel over the result of a calculation. (This is an important fact.) – Were it otherwise: if, for instance, one mathematician was convinced that a figure had altered unperceived, or that his or someone else's memory had been deceptive, and so on, – then our concept of "mathematical certainty" would not exist.

The *Entscheidungsproblem*

Show that there exists a definite method that can determine, for every statement of mathematics expressed formally in an axiomatic system (using first-order logic), whether or not that statement can be deduced from the axioms.

What *is* a "Definite Method"?

To satisfactorily resolve the *Entscheidungsproblem* one must:

- Analyze what is meant *in general* by a "formal system" and a "step" in a formal system in the relevant Hilbertian sense.
- This could not be done by simply writing down another formal system or by discussing in the metalanguage various kinds of different formal systems.
- This is why the ("logic-free" versions of) λ-definability and the Herbrand-Gödel-Kleene equational systems were used, and also why Turing devised his machines with command-tables.

Negative Resolution of the *Entscheidungsproblem*, Co-extensiveness

- Church, Kleene and Rosser (1935) showed that the class of functions calculable in the Herbrand-Gödel-Kleene equational calculus is co-extensive with the class of λ-definable functions.
- Church (1935-36), building on Gödel (1931), demonstrated that there is no "effectively calculable" function which decides whether two λ-definable expressions are equivalent.
- Turing (1936) showed that no "machine" can "compute" the desired general procedure as an "application" of his wholly novel analysis. Appendix: the functions his "machines" can "compute" are just those that are λ-definable.

Turing (1936)

- Turing's particular way of resolving the *Entscheidungsproblem* was *not* the application of a preexisting blueprint of ideas and methods in the metamathematics literature.
- Rather, Turing offered a philosophically informed, analytic exercise. An intuitively satisfying survey of...surveyability!
- Turing's deployment of his central argument bears the stamp of Wittgenstein's way of thinking about logic "anthropologically", rather than "metamathematically".

Turing's Analysis

Turing analyzed what a "step" in a formal system *is* by thinking through what it is *for*, i.e., what is *done* with it.

The comprehensiveness of his treatment—its lack of "morals"—lies here.

Turing made the very idea of a formal system *plain*, or "homespun" [Wittgenstein's term].

Wittgenstein: "Turing's 'Machines'. These are *humans* who calculate".

Wittgenstein on "Surveyability" 1937-39

Surveyability is part of proof (RFM I §154).

'A mathematical proof must be surveyable' (RFM III §1).

We construct the proof once and for all (RFM III §22).

Ideas:

- *Principia* proofs need to be *made* surveyable with the help of a variety of different mathematical techniques.
- Communicability is a central feature of proof: the "calculational" aspect requires that disagreements terminate in agreement that a conclusion *follows*.

Mühlhölzer 2006, 2010: Wittgenstein on "Surveyability"

- The surveyability of a proof consists in its possibility of [exact] reproduction.
- This reproduction must be an easy task.
- We must be able to decide with certainty whether the reproduction produces the same proof.
- The reproduction of a proof is of the sort of a reproduction of a picture.
- "Surveyable" does not imply mathematical understanding.

Main Hilbertian Points for Our Purposes

- Symbolic prostheses are a precondition of the application of logic and mathematics. (Parameters.)
- These symbols are extra-logical, discrete, and intuitively immediate before all thought.
- These symbols are irreducible.
- Logic's and mathematics' certainty depends upon the surveyability of these symbols in all their parts (simplicity).
- Surveyability involves *communicability* and termination of disputes, i.e., resolution of differences with certainty.
- Hilbert is not a "formalist" about the *content* of mathematics.

Turing's Machines (1936): Everyday marks of the concept "calculation"

- A human computor works locally, step-by-step, and can only take in a certain number of symbols "at a glance". [Hilbert]
- The computor takes in "simple operations ... so elementary that it is not easy to imagine them further divided". [Hilbert]
- We "avoid introducing the notion of a 'state of mind' by considering a more physical and definite counterpart: it is always possible for the computor to break off from his work, to go away and forget all about it, and later to come back and go on with it. If he does this he must leave a note of instructions (written in standard form) explaining how the work is to be continued. This note is the counterpart of the 'state of mind' ". [Wittgenstein]

Turing Offers a “Language-Game”, not a Thesis about *Computability*

Turing (1936) argues

1. By “intuition”.
2. By showing the set of λ-calculable functions is co-extensive with the set of Turing computable ones.
3. By giving examples of “computable” real numbers.

Argument form: Suppose that what a human computation is, in general, is something like *this*. Then how could the procedures followed *not* yield “surveyability”?

Hilbertian Features of Turing's Analysis

- Turing crafts his particular diagonal argument in "On Computable Numbers" carefully, so that even an intuitionistic logician who rejects the law of the excluded middle in infinite contexts can accept his proof, as well as his analysis of the idea of a "step" in a formal system.
- It is not part of our notion of "following a rule step-by-step" that we do or do not obey the law of excluded middle.
- The human interface, the human context of a shareable command, is *demonstrated* to be fundamental to the nature of computation.
- Turing's analysis of a "step" in a formal system is (and must be) altogether independent of *which* formal system we are speaking of, or *which* "states of mind" are actually used.

Gödel 1946 Praises Turing's Analysis

In all other cases treated previously, such as demonstrability or definability, one has been able only to define them relative to a given language, and for each individual language it is clear that the one thus obtained is not the one looked for. For the concept of computability, however, although it is merely a special kind of demonstrability or definability, the situation is different. By a kind of miracle it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion.

Gödel 1964

- "The precise and unquestionably adequate definition of the general concept of formal system [made possible by Turing's work allows the incompleteness theorems to be] proved rigorously for every consistent formal system containing a certain amount of finitary number theory."
- "With Turing's analysis of computability one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion, i.e., one not depending on the formalism chosen."

Wittgenstein 1939 (Turing in the audience)

- "What is a 'representative piece of the application'? ... Suppose I say to Turing, 'This is the Greek letter sigma', pointing to the sign σ. Then when I say, "Show me a Greek sigma in this book", he cuts out the sign I showed him and puts it in the book. –Actually these things don't happen."
- Don't treat your common sense like an umbrella. When you come into a room to philosophize, don't leave it outside but bring it in with you.
- Mathematical "techniques" discussed 114 times.

Turing on Phraseology: "The Reform of Mathematical Notation" (1942-44)

The statement of the type principle given below was suggested by lectures of Wittgenstein, but its shortcomings should not be laid at his door.

Turing, "The Reform of Mathematical Notation" (1942-44)

Symbolic logic is a very alarming mouthful for most mathematicians, and the logicians are not very much interested in making it more palatable. It seems however that symbolic logic has a number of small lessons for the mathematician which may be taught without it being necessary for him to learn very much of symbolic logic.

In particular it seems that symbolic logic will help the mathematicians to improve their notation and phraseology.

Turing: "The Reform of Mathematical Notation" (1942-44)

We should conduct an extensive examination of current mathematical, physical and engineering books and papers with a view toward listing all commonly used forms of notation and examine them to see what they really mean. This will usually involve statements of various implicit understandings as between writer and reader. But the laying down of a code of minimum requirements for possible notations should be exceedingly mild, avoiding the straightjacket of a logical notation.

Turing: The Reform of Mathematical Notation" (1942-44)

It would not be advisable to let the reform [of notation] take the form of a cast-iron logical system into which all the mathematics of the future are to be expressed. No democratic mathematical community would stand for such an idea, nor would it be desirable.

But what about Turing and AI?

Turing, "Intelligent Machinery" (1948): The "Intellectual" Search

We might arrange to take all possible arrangements of choices in order, and go on until the machine proved a theorem which, by its form, could be verified to give a solution of the problem ... Further research into intelligence of machinery will probably be very greatly concerned with "searches" of this kind. We may ... call such searches "intellectual searches".

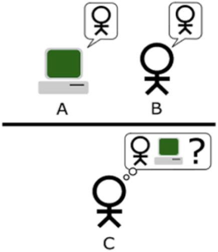
Turing (1948): The Evolutionary Search

It may be of interest to mention two other kinds of search in this connection. There is the genetical or evolutionary search by which a combination of genes is looked for, the criterion being survival value. The remarkable success of this search confirms to some extent the idea that intellectual activity consists mainly of various kinds of search.

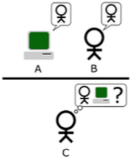
Turing (1948): The Cultural Search

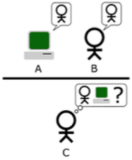
The remaining form of search is what I should like to call the “Cultural Search‘... [T]he isolated man does not develop any intellectual power. It is necessary for him to be immersed in an environment of other men, whose techniques he absorbs during the first 20 years of his life. He may then perhaps do a little research of his own and make a very few discoveries which are passed on to other men. From this point of view the search for new techniques must be regarded as carried out by the human community as a whole, rather than by individuals.

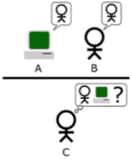
The Turing Test for “Intelligent” Machinery (1950)



The Turing Test as a Social Experiment in Phraseology







What Turing is *Not* Doing with the Turing Test

- Trying to prove that machines *can* think.
- Assuming that behaviorism is true.
- Trying to prove that machines are conscious and capable of emotion.
- Trying to explain or deny the fact of consciousness.
- Trying to prove that humans are machines.
- Trying to prove that machines are indistinguishable from humans.
- Merely stipulating an operational or behavioristic definition of "intelligence".
- Assuming that disinterpreted operations with signs are capable of grounding meaning.

What Turing *Is* Doing with the Turing Test

- Showing that one cannot prove a negative result – e.g., that machines *cannot* think – because as yet one does not have a clear enough concept of "thought".
- Showing us how we might explore together the "emotional" effects of computational machinery on our ways of expressing ourselves.
- Framing a repeatable, social, philosophically-minded human-to-human experiment in phraseology, or ordinary language, *in life*.
- Allowing us to make *surveyable* our concept of "thinking".

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